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ON THE EXISTENCE OF LINEAR EQUILIBRIA IN MODELS OF MARKET MAKING

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We derive necessary and sufficient conditions for a linear equilibrium in three types of competitive market making models: Kyle type models (when market makers only observe aggregate net order flow), Glosten–Milgrom and Easley–O'Hara type models (when market makers observe and trade one order at a time), and call markets models (individual order models when market makers observe a number of orders before pricing and executing any of them). We study two cases: when privately informed (strategic) traders are symmetrically informed and when they have differential information. We derive necessary and sufficient conditions on the distributions of the random variables for a linear equilibrium. We also explore those features of the equilibrium that depend on linearity as opposed to the particular distributional assumptions and we provide a large number of examples of linear equilibria for each of the models.

Key Words: market microstructure, market making, strategic trading, linear equilibria

1. INTRODUCTION

Models of market making are used to understand the effects of different exchange structures on market depth, price volatility, informativeness of prices, and the ability of informed traders to exploit their private information.[1] They are also important building blocks for the study of other questions whose answers depend, in part, on what happens in the market for the firm's asset.[2] Most of these analyses focus on a closed-form, usually linear, solution to the embedded model of stock trading. In this paper, we seek

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[1] For excellent surveys, see Easley and O'Hara (1995), Black (1992), or O'Hara (1995).

[2] Examples include the growing literature on the social impact of insider trading (e.g., Fishman and Hagerty 1992; Leland 1992; Khanna, Slezak, and Bradley 1994) and models of price manipulation (e.g., Allen and Gale 1992; Kumar and Seppi 1992; Bagnoli and Lipman 1996).

necessary and sufficient conditions for a linear equilibrium in most standard models of market making with risk neutral agents and independent random variables. This approach is similar to that of Ross (1978) who provides necessary and sufficient conditions on distributions for the mutual fund separation theorem and to that of Chamberlain (1983) who provides necessary and sufficient conditions on distributions so that preferences can be characterized by mean-variance utility functions, meaning that asset prices follow the CAPM.

Most models in the market microstructure literature are either Kyle type models that follow Kyle (1985) or Glosten–Milgrom–Easley–O'Hara (henceforth GMEO) type models that follow Easley and O'Hara's (1987) extension of the original Glosten and Milgrom (1985) model. In formulations that follow Kyle (1985, 1986a, 1986b), market makers observe aggregate net order flow and then set prices. All closed-form solutions are linear and virtually all assume that the random variables are independent and normally distributed (the exception being Foster and Viswanathan 1993, who study elliptically distributed but not independent random variables). Most results concerning the effect of exchange structure and most analyses based on Kyle's model rely on the implications from these linear equilibria.

Given this background, our paper has two aims. Our first aim is to find necessary and sufficient conditions for linear equilibria in Kyle type models without imposing the assumption of normality.[3] Next, we will show that many comparative statics derived in previous work (e.g., regarding price sensitivity, the number of strategic traders, and the variability of exogenous trading) derive from the linearity of equilibrium and not from any primitive distributional assumptions.

Our second aim is to find general conditions for linear equilibria in other models of market making. Delineating conditions under which linear equilibria can obtain in other market structures may help us understand differences between market structures and possibly lead to alternative models of market structures. We consider the issue of the existence of linear equilibrium in two alternative market structures: (i) the GMEO formulation in which traders arrive and trade one at a time and (ii) a model of call markets where market makers observe individual orders but not the identity of the individual submitting the order (henceforth IO).[4] From a practical standpoint, this formulation is closer to real world markets than the commonly used Kyle approach.[5]

We employ a game-theoretic model that has all of these formulations as special cases. There are risk-neutral, privately informed traders who submit market orders. Exogenous traders submit market orders for unmodeled reasons, represented by a vector of random variables, $\vec{u}$, *not* taken to be independent. There are also risk-neutral market makers who observe something about the vector of submitted orders and then compete by choosing prices at which they will execute the orders.[6] The standard formulations are obtained by making different assumptions on what market makers observe about the submitted orders before they price them. Kyle type models are obtained by assuming that market

[3] In a paper subsequent to ours but independent of our work, Noldeke and Troger (1998) derive some similar results. They obtain a result similar to our Theorem 3.1 and a somewhat stronger result than Theorem 3.5.

[4] Corb (1993) independently analyzed equilibria in this type of model, specifically focusing on nonlinear equilibria.

[5] Models of market making in which the risk-neutral strategic trader observes the liquidity traders' orders prior to submitting her own have been studied in Kyle (1989), and Kyle and Vila (1991). Rochet and Vila (1994) provide conditions for existence and uniqueness of equilibrium in such models. Since the strategic trader's problem is so different in such models, their work is very different but complementary to ours.

[6] Note that because we restrict our attention to risk-neutral traders and market makers, we leave open the question of whether there are other combinations of preferences and distributions that support linear equilibria.

makers observe aggregate net order flow (the order imbalance), choose prices, and then execute *every* order. GMEO type models are obtained by assuming that market makers observe one order (but do not know whose), price that order, and then execute only that order. The IO model is obtained by assuming that market makers observe every order submitted, choose prices, and then execute every order.

We provide necessary and sufficient conditions on probability distributions for a linear equilibrium in all three formulations when strategic traders see the same signal (are symmetrically informed). When strategic traders are differentially informed, we obtain necessary and sufficient conditions for GMEO and IO type models but only sufficient conditions for Kyle type models.[7] We show that the conditions also depend on whether the number of traders who have private information and submit orders is exogenous. In all cases, necessary and sufficient conditions can be expressed either in terms of the induced probability distribution of strategic and exogenous traders' orders or in terms of the underlying distribution of strategic traders' signals and the distribution of exogenous traders' orders. The former usually provides a much more intuitive version but suffers from the defect that it is not in terms of exogenous distributions.

In Kyle type models, with N symmetrically informed strategic traders, we show that there is a linear equilibrium if and only if the aggregate order from strategic traders has the same distribution as the sum of N i.i.d. random variables each equal in distribution to u, the exogenous traders' aggregate net order. We use these conditions to indicate which results depend on the restriction to a linear equilibrium and which depend on the normality assumptions. We show that if the distributions admit a linear equilibrium, there is only one and the intercept and slope depend only on the first two moments of the distributions (when they exist).[8] Thus, for a fixed number of strategic traders, the results in the literature do not depend on the normality assumptions at all. Also, we provide examples where different combinations of distributions lead to the *same* linear equilibrium pricing equation and compute a wide variety of examples compiled in Table 3.1. Unfortunately, results are less robust when there must be a linear equilibrium for an endogenous number of strategic traders—a linear equilibrium for every possible number of strategic traders. As we show, this implies that the signals observed by strategic traders, s and u, must be stable random variables[9] or normal if one also requires finite second moments.[10]

We also derive sufficient conditions when strategic traders see different signals. There is no nice interpretation but we show how to use these sufficient conditions to compute examples using the class of stable distributions. We find that the normal example behaves

[7] Essentially, in Kyle type models, when strategic traders are differentially informed, they do not know what signals other strategic traders saw. As a result, they play a game of incomplete information even if the price is a linear function of order flow. In such circumstances, we can find an equilibrium in which each strategic trader's order is a linear function of his private information but cannot guarantee that there are not other sets of trading strategies that would support a linear price function.

[8] We also provide examples with stable random variables, a class introduced to finance by Mandlebrot (1963) and Fama (1963) as alternative distributions for stock returns. (Their work was extended by McCulloch 1978 who showed how to handle continuous time processes with stable increments.) Other than the normal, these distributions do not have finite second moments and play a major role later in our analysis.

[9] The class of stable distributions is the set of limiting distributions for normed sums of i.i.d. random variables. The Normal, Pareto, Lévy, and Cauchy distributions are examples of stable distributions.

[10] If strategic traders make all decisions after learning their private information (such as when they weigh the benefits from trading against the costs of violating insider trading laws), finite second moments are not needed. However, if strategic traders make decisions prior to learning their private information (such as when deciding whether to acquire private information) their objective function is not well defined unless second moments exist. Since the normal distribution is the only stable distribution with finite second moments, one requires normality to obtain a linear equilibrium in these cases.

differently than the other stable distribution examples. With normality, there is a linear equilibrium for any number of strategic traders, they trade more aggressively the more of them there are and the slope of the price function declines in the number of strategic traders. If the random variables are stable but not normal, there are a maximum number of strategic traders for which a linear equilibrium exists.

In GMEO type models, necessary and sufficient conditions for a linear equilibrium are much simpler: (i) each strategic trader's order must have the same distribution as each exogenous trader's order *and* (ii) the probability that the order executed is submitted by a strategic trader is one-half.[11] The first condition implies that the existence of a linear equilibrium does not depend on whether strategic traders are symmetrically or differentially informed because only one strategic trader's order is executed. The second condition implies that the existence of a linear equilibrium does not depend on whether the number of strategic traders is endogenous or exogenous unless one ties the probability that the order is placed by a strategic trader to the proportion of strategic traders. Again, we show that if linear equilibrium exists, there is only one, and we characterize the intercept and slope. We also provide examples of different combinations of distributions that lead to the same price function and compute Table 4.1 for the same set of examples as we did for Kyle type models.

In IO type models, we show that any distributional conditions that support a linear equilibrium in an IO type model support one in Kyle type models because it is assumed that market makers learn nothing from the position of an order in the order vector. However, some distributions that support linear equilibria in Kyle type models will not support linear equilibria in IO models. Further, our results for IO type models depend heavily on whether strategic traders can split their orders. If they cannot, then there is a linear equilibria in an IO model if and only if there is one strategic and one exogenous trader and their orders have the same distribution and then our results for Kyle and GMEO type models carry over to IO type models.[12] We also derive necessary and sufficient conditions for a linear equilibrium in IO type models for one strategic trader when she is permitted to submit multiple orders.[13] We show that there is a linear equilibrium if and only if the strategic trader's total order has the same distribution as the aggregate net order of the exogenous traders—exactly the same condition as we obtained in Kyle type models and so all of our results for Kyle type models carry over in this case.

Finally, we note that techniques can be applied to other areas where linear equilibria are commonly used. An example would be the literature in industrial organization dealing with information sharing that relies heavily on linear conditional expectations.[14] Our

[11] Intuitively, the first condition arises because only one order is executed and it must either be placed by a strategic trader or by an exogenous trader. The second condition is less obvious. Intuitively, if it is more likely (less likely) than not that the order is placed by a strategic trader, then the market makers' optimal price adjustment (the slope of the price function) to any realized order flow is always greater than (less than) that conjectured by the strategic trader because it is more likely (less likely) that the order is informative. Thus, it is only when the order is equally likely to have been placed by a strategic trader or an exogenous trader that market makers adjust price by exactly what was conjectured by the strategic trader.

[12] With one strategic and one exogenous trader, conditions for a linear equilibrium are the same as in Kyle type models, which implies that all of our results on such equilibria carry over to the linear equilibria in IO models. Further, if the probability that a strategic trader's order is executed equals the fraction of strategic traders, then the restriction to one strategic and one exogenous trader makes the linear equilibria in Kyle, GMEO, and IO type models the same.

[13] We restrict attention to one strategic trader because, otherwise, when strategic traders can split their orders, the analysis of how they split them in equilibrium takes a prominent role in the analysis. Since it does not in any of the other models, we leave this for future work.

[14] See, for example, Vives (1989) or Hwang (1993) and the references therein.

approach provides a general method to characterize distributions that yield linear conditional expectations and simultaneously satisfy the conditions for a linear equilibrium implied by the theory.

The remainder of the paper is organized as follows. Section 2 is a description of the general model. Section 3 specializes it to Kyle type models of market making, Section 4 deals with GMEO type, and Section 5 with individual order type models. Section 6 contains our conclusions. Most proofs are in the Appendix.

2. THE MODEL

There are three types of risk neutral agents: exogenous traders, strategic traders, and market makers. We assume that all random variables have finite expectations in order to evaluate the agents' objective functions. The agents interact in a single-period market for a single traded asset whose terminal value is v per share. The market is described by the following order of play. Exogenous and strategic traders submit market orders without knowing the orders submitted by other traders. There are no restrictions on short sales and orders can be positive or negative.[15] Then, market makers observe something about the submitted orders and choose prices. They are required to stand ready to execute any or all of the order flow at their chosen prices. The orders are then executed at the most favorable price for the traders. This game-theoretic structure has as special cases Kyle type, GMEO type, and the individual order models that we wish to analyze.[16] Which version is studied is determined by what market makers observe about the submitted orders prior to pricing them.

More specifically, in the model there are L exogenous traders who have no private information and who place market orders for unmodeled reasons. Hence each exogenous trader's order is a random variable u_i, for $i = 1, 2, \dots, L$. Here u_i represents the number of shares demanded by the ith exogenous trader. Let $u = \sum_{i=1}^{L} u_i$ be the total orders from exogenous traders. There are $N \geq 1$ strategic traders each of whom receive a signal (their private information about the terminal value of the asset), s_i (which need not be v and may not be identical across strategic traders). Based on s_i, each strategic trader submits a market order x_i (i.e., he cannot condition on the price at which the trade will execute). Both the exogenous trader orders and strategic trader orders are placed in the first stage. Finally, there are $K \geq 2$ market makers who act like "Bertrand" competitors. In the second stage, these market makers observe some information about the order flow and base their prices on this information. We assume that for all j, u_j is independent of $(v, s_1, \dots, s_N)$.

We have a game of incomplete information because market makers do not know the signals strategic traders receive and so we seek Bayes–Nash equilibria. We focus on linear equilibria because we wish to characterize necessary and sufficient conditions for a linear Bayes–Nash equilibrium. By linear, we mean market makers choose price functions that are linear in what they observe about the submitted orders. Finally we require that the Bayes–Nash equilibrium be sequentially rational, sometimes referred to as a Perfect Bayes–Nash equilibrium.[17]

[15] We adopt the convention that if the order is positive, the trader wishes to buy and, if negative, wishes to sell shares.

[16] See also Bagnoli and Holden (1994).

[17] For additional details, see Fudenberg and Tirole (1991) or Gibbons (1992).

3. KYLE TYPE MODELS

Here, market makers observe aggregate net order flow; that is, if $(x_1, \ldots, x_N, u_1, \ldots u_L)$ is the vector of submitted orders, then market makers observe $z \equiv \sum_{i=1}^{N} x_i + u$. Initially we will consider models where the N strategic traders observe the same information $s = s_i \, \forall i$. Then we will consider models where the N strategic traders observe differential information—that is, the s_i and s_j are different for traders i and j.

3.1. An Exogenous Number of Symmetrically Informed Strategic Traders

The number of strategic traders, N, is exogenously given. All strategic traders observe the same signal $s_i = s$, where $\mathrm{E}[v \mid s] = s$, prior to submitting their market orders.[18] With a two-stage game, one starts at the last stage and works backward to find sequentially rational Bayes–Nash equilibria. In the last stage, risk-neutral market makers select prices to maximize their expected profits given their observation of the realized aggregate net order flow z. A market maker's expected profit is $\mathrm{E}[(p - v)\frac{z}{k} \mid z]$, if he is one of the $k \leq K$ market makers quoting the most favorable price for traders. If aggregate net order flow is positive (negative), then a market maker executes his portion of it when his ask is lowest (bid is highest). Using the standard proof technique for "Bertrand" games, the unique Nash equilibrium is for market makers to offer $p(z) = \mathrm{E}[v \mid z]$ for equilibrium values of z. Because market makers condition on z, they use the strategies chosen by strategic traders and so this identification of the price charged equaling the conditional expected value of a share embeds a strategy function for each strategic trader.

Even though market makers infer (correctly, in equilibrium) the strategic traders' strategies, their aggregate order, X, is still random because market makers do not observe the private signal s that is known only to the strategic traders. If the support of X and/or the support of u permits aggregate net order flow to be any real number, then every conceivable value of aggregate net order flow can arise on the equilibrium path—there are no off-the-equilibrium paths.[19]

Because we seek necessary and sufficient conditions for a linear equilibrium, we seek conditions such that each market maker's best reply is linear in z with a positive slope. Given this, strategic traders infer that[20] $p(z) = \mu_N + \lambda_N z$. The ith strategic trader's problem is

$$\max_{x_i} \mathrm{E}[(v - p(z))x_i \mid s] \equiv \max_{x_i} (s - \mu_N - \lambda_N(x_i + X_{\sim i} + \mathrm{E}[u]))x_i,$$

where x_i is the order she places, u is the exogenous traders' aggregate net order, and $X_{\sim i}$ is the sum of the orders placed by other strategic traders. Her objective function was simplified by noting that $\mathrm{E}[v \mid s] = s$ and $\mathrm{E}[u \mid s] = \mathrm{E}[u]$ because u is independent of s. The first-order condition is

$$x_i = \frac{s - \mu_N - \lambda_N(X_{\sim i} + \mathrm{E}[u])}{2\lambda_N}. \tag{3.1}$$

The second-order condition is met if $\lambda_N > 0$, which we have imposed.

[18] This is without loss of generality. In the more general case where $\mathrm{E}[v \mid s] = ks$ we can rescale the signal s so that $k = 1$.

[19] If there are, then we must specify off-the-equilibrium path beliefs for market makers. When the support of the distribution is an interval (the case considered in this paper), we assume that if the market makers observe an off-the-equilibrium aggregate net order, then they set prices so that strategic traders never find it optimal to buy (sell) at these prices. Extensions to cases where the support of the distribution is not an interval are beyond the scope of the current paper.

[20] We subscript μ and λ to indicate that, in Kyle type models, market makers know the number of strategic traders and so the equilibrium price function can depend on N.

Since strategic traders are symmetrically informed, we can compute Bayes–Nash equilibria by solving for Nash equilibria. Summing the first-order conditions and solving show that there is a unique Nash equilibrium in which strategic traders choose the same, pure strategy[21]

$$x(s) = \frac{s - \mu_N - \lambda_N \mathrm{E}[u]}{(N+1)\lambda_N} \equiv \beta_N(s - \mu_N - \lambda_N \mathrm{E}[u]). \tag{3.2}$$

Note that each strategic trader's order is a linear function of her private information. Hence, when computing the market makers' conditional expected value of the asset given z, we must impose that the conjectured strategy for strategic traders is linear in s. If we do not, then we will not have an equilibrium. Thus, we seek necessary and sufficient conditions for the market makers' conditional expectation, $\mathrm{E}[v \mid z]$ to be linear when (3.2) describes a strategic trader's order. These conditions can be found by applying a lemma from Ferguson (1958, Lem. 1).[22]

FERGUSON'S LEMMA. *For jointly distributed random variables $Y_0, Y_1, \ldots, Y_n$, each with mean zero, the following are equivalent:*

(i) $\mathrm{E}[Y_0 \mid Y_1, Y_2, \ldots, Y_n] = \sum_{i=1}^{n} b_i Y_i \quad$ a.e.,

(ii) $\frac{\partial}{\partial t_0}\phi_{Y_0,Y_1,\ldots,Y_n}(t_0, t_1, \ldots, t_n)\Big|_{t_0=0} = \sum_{i=1}^{n} b_i \frac{\partial}{\partial t_i}\phi_{Y_1,\ldots,Y_n}(t_1, \ldots, t_n)$,

where ϕ_w is the characteristic function for the random vector w.[23]

Since the lemma is for mean zero random variables, we transform ours by subtracting their means, denoted by a " ˆ " above the random variable. Since we assumed that $\mathrm{E}[v \mid s] = s$, then $\mathrm{E}[v] = \mathrm{E}[s]$ and so $\mathrm{E}[v \mid z]$ is linear if and only if $\mathrm{E}[\hat{v} \mid \hat{z}] = \lambda_N \hat{z}$. Further, using the fact that the joint distribution of v, s, z can be written as a product of the joint distributions of v, s and s, z, one can show that $\mathrm{E}[v \mid z] = \mathrm{E}[s \mid z]$. Hence, we seek necessary and sufficient conditions for $\mathrm{E}[\hat{s} \mid \hat{z}] = \lambda_N \hat{z}$.

Toward this end, we first impose the following technical condition.[24] In all of our discussion, we will assume that this condition holds.

Technical Condition. Suppose there are N strategic traders. The characteristic function of $\hat{s}$ is strictly zero outside the interval (−U,U) (where U is possibly ∞) and satisfies the following smoothness restrictions on (−U,U): If $\phi_{\hat{s}}(t)$=0, then $\phi_{\hat{s}}() \in C^N$ in an open neighborhood of t and $\phi_{\hat{s}}^i(t) \neq 0$ for $i = 1, \ldots, N$. An identical condition is satisfied by $\hat{u}$.

[21] Note that we are not requiring or assuming that strategic traders adopt the same trading strategy. We are showing that the unique equilibrium has the strategic traders *choosing* the same trading strategy given that the equilibrium price function is a linear function of the aggregate net order flow.

[22] He describes his lemma as corresponding to a lemma in Fix (1949). See also Blake and Thomas (1968).

[23] The characteristic function is an alternative way to characterize a random variable's distribution function. There are standard inversion formula that allow one to obtain one from the other (see Lukacs 1960 for details). In our context, the characteristic function of $Y_0, Y_1, \ldots, Y_n$, $\phi_{Y_0,Y_1,\ldots,Y_n}(t_0, t_1, \ldots, t_n) = E[it \cdot Y]$, where $t \cdot Y \equiv \sum_{j=0}^{n} t_j Y_j$.

[24] We thank Georg Noldeke and Thomas Troger for bringing this issue to our attention.

The technical condition above restricts the characteristic function to be continuously differentiable of order N and requires nonzero derivatives up to order N when the characteristic function is zero.[25] Without this smoothness condition, one can create examples that violate Theorem 3.1 below. We provide one such example in the Appendix. However, this example is somewhat pathological (it does not correspond to any standard distribution function). Most distributions that are considered in practice satisfy the smoothness condition that we impose.

In independent work, Noldeke and Troger (1998) impose the technical condition that the moment generating function of the relevant random variables exists in a neighborhood of 0. This implies that all the moments exist and are finite (see Billingsley 1979, Sec. 30) and that the behavior of the characteristic function around zero completely characterizes the random variable. From a standard theorem (e.g., see Renyi 1970), if the first n moments exist, then the characteristic function is differentiable n times. Hence, Noldeke and Troger's condition assumes that the characteristic function is infinitely differentiable.[26]

Definition of N-fold average. A random variable y_1 is an N-fold average of another random variable y_2 if y_1 has the same distribution as the average of N independent random variables each with the same distribution as y_2.

Theorem 3.1. *For fixed N, there is a linear equilibrium if and only if $\hat{x}$ and $\hat{u}$ have finite expectations and $\hat{x}$ is an N-fold average of $\hat{u}$.*[27]

We use (3.2) to provide conditions on the exogenous distributions of s and u and complete the analysis by solving for μ_N and λ_N.

Theorem 3.2. *For fixed N, there is a linear equilibrium if and only if $\hat{s}$ and $\hat{u}$ have finite expectations and $\hat{s}$ has the same distribution as $(N+1)\lambda_N$ times the N-fold average of $\hat{u}$.*

Proof. From Theorem 3.1, for all t, $\phi_{\hat{u}}(t)^N = \phi_{N\hat{x}}(t) = \phi_{\hat{x}}(Nt) = \phi_{\hat{s}}((N/\lambda_N(N+1))t)$, since $\hat{x} = \hat{s}/((N+1)\lambda_N)$. Thus, $\hat{s}$ has the same distribution as $(N+1)\lambda_N$ times the N-fold average of $\hat{u}$. □

Theorem 3.3. *For fixed N, there is only one linear equilibrium. Its intercept is* $\mathrm{E}[s] - \lambda_N \mathrm{E}[u]$ *and its slope is* $\lambda_N = \frac{1}{(N+1)\beta_N}$, *which is uniquely characterized by*

$$\phi_s(N\beta_N t) = e^{iN(\beta_N \mathrm{E}[s] - \mathrm{E}[u])}\phi_u(t)^N \qquad \forall t. \tag{3.3}$$

If the distributions have finite second moments,

$$\lambda_N = \frac{\mathrm{Cov}[s,z]}{\mathrm{Var}[z]} = \frac{1}{N+1}\sqrt{\frac{N\sigma_s^2}{\sigma_u^2}},$$

where $\sigma_s^2 \equiv \mathrm{Var}[s]$ *and* $\sigma_u^2 \equiv \mathrm{Var}[u]$.

[25] More generally, if the characteristic function and its first K derivatives are zero, we need that the $K+1$th to the $K+N$th derivatives are nonzero and that the $K+N$th derivative is continuous.

[26] Noldeke and Troger's condition has the advantage of being independent of N, the number of strategic traders.

[27] Most proofs are in the Appendix.

Our theorems provide four insights into the linear equilibria. First, if the distributions of the signal and exogenous trades have finite second moments, only their first two moments affect the equilibrium price function. As a result, we will be able to find combinations of distributions that lead to the *same* price function.[28] Second, in standard analyses with normality, one usually sees a proof that there is a unique linear equilibrium. Theorem 3.3 indicates that if there is a linear equilibrium, then there is only one—normality is unimportant. Third, the intercept, μ_N, is not, in general, independent of N unless the expected value of the exogenous traders' aggregate net orders is zero. Finally, just as with normality, we find that, for distributions with finite second moments, λ_N falls and $N\beta_N$ rises in N—normality is unimportant.

We close this section by considering a standard special case: each exogenous trader's order is independent and identically distributed.

COROLLARY 3.1. *If there are N strategic traders and the u_i's are i.i.d., then there is a unique linear equilibrium if and only if $\hat{x}$ and $\hat{u}$ have finite expectations and $\hat{x}$ is L times the NL-fold average of $\hat{u}_i$. Further, if the distributions have finite second moments, $\hat{s}$ has the same distribution as $\sqrt{NL}(\frac{\sigma_s}{\sigma_{u_i}})$ times the NL-fold average of $\hat{u}_i$.*

Proof. Since $\hat{u}=\sum_{i=1}^{N} \hat{u}_i$ and the $\hat{u}_i$ are identically distributed, using Theorem 3.1, we obtain the required result. Using Theorem 3.2 then yields the second claim. □

3.1.1. *Examples of Linear Equilibria.* First, we explain how Theorems 3.1, 3.2, and 3.3 are used to create Table 3.1, which characterizes linear equilibria for different combinations of distributional assumptions. We illustrate with the well-known example of normally distributed random variables. Let $u \sim \mathcal{N}(\alpha, \sigma_u^2)$ and $s \sim \mathcal{N}(\xi, \sigma_s^2)$ for means $\alpha, \xi \in \mathcal{R}$ and variances $\sigma_s^2, \sigma_u^2 \in \mathcal{R}_+$. The characteristic function of $y \sim \mathcal{N}(a, b)$ is $\phi_y(t) = e^{ita-(t^2b/2)}$. Substituting into (3.3) and suppressing the subscripts on β and λ, one obtains $e^{i(N\beta t)^2\sigma_s^2} = e^{iN\sigma_u^2 t^2}$ for all t which can be true only if $(N\beta)^2\sigma_s^2 = N\sigma_u^2$. Since $\beta = 1/(\lambda(N+1))$, we obtain the well-known result that for normally distributed random variables,

$$\lambda_N = \left(\frac{1}{N+1}\right)\sqrt{\frac{N\sigma_s^2}{\sigma_u^2}}.$$

Using the characterization of the intercept in Theorem 3.3 yields the equilibrium price function. All of the other computations summarized in Table 3.1 proceed in an analogous way.[29]

Potentially the most important application of this approach is that it allows us to readily illustrate that many combinations of distributions produce the *same* linear equilibrium. To see this, pick a linear function, $\rho(z) = \mu + \lambda z$ for $\mu \in \mathcal{R}$, $\lambda > 0$. For each of the first four distributions in Table 3.1, Part A, we can select parameters for the distributions of s and u such that the linear equilibrium price function described in Table 3.1 is the same as ρ when $N = 1$. Similar exercises can be done for the remaining four distributions in Part A as well as for the two and the remaining five in Part B.

[28] A similar result has been derived by Foster and Viswanathan (1993) when (v, u) are in the elliptically contoured class of distributions.

[29] We postpone presenting examples with distributions that have no second moments until the next section. The reason is that the most straightforward examples rely on the distributions being stable and such distributions are central in the analysis of the next section.

TABLE 3.1
Kyle Type Models With Symmetric Information

Distribution	Distributions of u and s	N	Linear equilibrium price function
Part A: Continuous Distributions			
Normal: $y \sim \mathcal{N}(a, b)$	$u \sim \mathcal{N}(\alpha, \theta)$		
$y, a \in \mathcal{R}, b > 0$	$s \sim \mathcal{N}(\xi, \gamma)$	any	$p(z) = \xi + \left(\frac{1}{N+1}\sqrt{\frac{N\gamma}{\theta}}\right)(z - \alpha)$
$f(y) = \frac{1}{\sqrt{2\pi b}}\exp\left\{-\frac{(y-a)^2}{2b}\right\}$			
Laplace: $y \sim LP(a, b)$	$u \sim LP(\alpha, \theta)$		
$y, a \in \mathcal{R}, b > 0$	$s \sim LP(\xi, \gamma)$	$N = 1$	$p(z) = \xi + \left(\frac{\gamma}{2\theta}\right)(z - \alpha)$
$f(y) = \frac{1}{2b}\exp\left\{-\frac{\lvert y-a\rvert}{b}\right\}$			
Logistic: $y \sim LOG(a, b)$	$u \sim LOG(\alpha, \theta)$		
$y, a \in \mathcal{R}, b > 0$	$s \sim LOG(\xi, \gamma)$	$N = 1$	$p(z) = \xi + \left(\frac{\gamma}{2\theta}\right)(z - \alpha)$
$f(y) = \frac{\pi \exp\left\{\frac{\pi(y-a)}{b\sqrt{3}}\right\}}{b\sqrt{3}\left(1+\exp\left\{\frac{\pi(y-a)}{b\sqrt{3}}\right\}\right)^2}$			
Extreme value: $y \sim EV(a, b)$	$u \sim EV(\alpha, \theta)$		
$y, a \in \mathcal{R}, b > 0$	$s \sim EV(\xi, \gamma)$	$N = 1$	$p(z) = \xi + \frac{k\gamma}{2} + \left(\frac{\gamma}{2\theta}\right)(z - \alpha)$
$F(y) = \exp\{-\exp\{-(y-a)/b\}\}$			$k =$ Euler's constant
Gamma: $y \sim \mathcal{G}(a, b)$	$u \sim \mathcal{G}(\alpha, \theta)$		
$y \geq 0, a, b > 0$	$s \sim \mathcal{G}(N\alpha, \gamma)$	any	$p(z) = \frac{N}{(N+1)\gamma}(\alpha + \theta z)$
$f(y) = \exp\{-y\theta\}y^{\alpha-1}\theta^{\alpha}\Gamma(\alpha)^{-1}$			
Chi–squared: $y \sim \chi^2(a)$	$u \sim \chi^2(\alpha)$		
$y > 0,\ a \in \mathcal{I}_+$	$s \sim \chi^2(N\alpha)$	any	$p(z) = \frac{N}{N+1}(N\alpha + z)$
$f(y) = \exp\left\{\frac{-y}{2}\right\}y^{\left(\frac{\alpha-2}{2}\right)}2^{-\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}$			
Inverse Gaussian: $y \sim \mathcal{IG}(a, b)$	$u \sim \mathcal{IG}(\alpha, \theta)$		
$y, a, b > 0$	$s \sim \mathcal{IG}(\frac{\alpha}{N}, \theta)$	any	$p(z) = \left(\frac{1}{N+1}\right)(n\alpha + z)$
$f(y) = \exp\left\{-\frac{b(y-a)^2}{2a^2 y}\right\}(\sqrt{b/2\pi y^3})$			
Pearson Type III: $y \sim P3(a, b, c)$	$u \sim P3(\alpha, \theta, \delta)$		
$y \geq a \in \mathcal{R}, b, c > 0$	$s \sim P3(\xi, \gamma, N\delta)$	any	$p(z) = \xi$
$f(y) = \frac{((y-a)/b)^{c-1}\exp\{(a-y)/b\}}{b\Gamma(c)}$			$+\left(\frac{N\gamma}{(N+1)\theta}\right)(N\delta\theta - \alpha + z)$
Part B: Discrete Distributions and Continuous Distributions on Finite Intervals			
Uniform: $y \sim \mathcal{U}(a, b)$	$u \sim \mathcal{U}(\alpha, \theta)$		
$y \in [a, b],\ a, b \in \mathcal{R}$	$s \sim \mathcal{U}(\xi, \gamma)$	$N = 1$	$p(z) = \frac{\gamma+\xi}{2}$
$f(y) = 1/(b-a)$			$+\left(\frac{\gamma-\xi}{2(\theta-\alpha)}\right)\left(z - \frac{\theta+\alpha}{2}\right)$
Triangular: $y \sim \mathcal{T}(a, b)$	$u \sim \mathcal{T}(\alpha, \theta)$		
$y \in [-a+b, a+b]; a, b \in \mathcal{R}$	$s \sim \mathcal{T}(\xi, \gamma)$	$N = 1$	$p(z) = \gamma + \left(\frac{\xi}{2\alpha}\right)(z - \theta)$
$f(y) = \frac{a-\lvert y\rvert}{a^2}$			
Binomial: $y \sim \mathcal{B}(a, b)$	$u \sim \mathcal{B}(\alpha, \theta)$		
$y \in \{0, 1, \ldots, a\}, a \in \mathcal{I}_+, b \in [0, 1]$	$\tau \sim \mathcal{B}(N\alpha, \theta)$	any	$p(z) = \left(\frac{N}{N+1}\right)(N + 1 - k\alpha\theta)$
$f(y) = \binom{a}{y}b^a(1-b)^{a-y}$	$s \equiv k\tau;\ k \in \mathcal{I}_+$		$+\left(\frac{Nk}{N+1}\right)z$
Negative binomial: $y \sim \mathcal{NB}(a, b)$	$u \sim \mathcal{NB}(\alpha, \theta)$		
$y \in \mathcal{I}_+, a \in \mathcal{I}_+, b \in (0, 1]$	$\tau \sim \mathcal{NB}(N\alpha, \theta)$	any	$p(z) = \left(\frac{Nk}{N+1}\right)\left(z + \frac{N\alpha(1-\theta)}{\theta}\right)$
$f(y) = \binom{y+a-1}{y}(1-b)^y b^a$	$s \equiv k\tau;\ k \in \mathcal{I}_+$		

TABLE 3.1
Continued...

Distribution	Distributions of u and s	N	Linear equilibrium price function
Pascal: $y \sim \mathcal{P}(a, b)$	$u \sim \mathcal{P}(\alpha, \theta)$		
$y, a \in \mathcal{I}_+, b \in [0, 1]$	$\tau \sim \mathcal{P}(\alpha, \theta)$	$N = 1$	$p(z) = \frac{k\alpha}{\theta} + \frac{1}{2}(z - \frac{\alpha}{\theta})$
$f(y) = \binom{a-1}{a-y} b^y (1-b)^{a-y}$	$s \equiv k\tau;\ k \in \mathcal{I}_+$		
Poisson: $y \sim P(a)$	$u \sim P(\alpha)$		
$y \in \mathcal{I}_+, a > 0$	$\tau \sim P(\alpha)$	$N = 1$	$p(z) = \frac{k}{2}(\alpha + z)$
$f(y) = a^y e^{-a}/y!$	$s \equiv k\tau;\ k \in \mathcal{I}_+$		
Geometric: $y \sim G(a)$	$u \sim G(\alpha)$		
$y \in \mathcal{I}_+, a \in [0, 1]$	$\tau \sim G(\alpha)$	$N = 1$	$p(z) = \frac{k}{2}\left(z + \frac{1}{\alpha}\right)$
$f(y) = a(1-a)^y$	$s \equiv k\tau;\ k \in \mathcal{I}_+$		

Not only does the approach outlined above simplify the task of computing a linear equilibrium but it also shows that many combinations of distributions support a linear equilibrium only if the number of strategic traders is one. In one section of Bagnoli and Holden (1994), uniform distributions are combined with risk neutral agents and a linear equilibrium is derived. They indicate that one parametric restriction is that there be exactly one strategic trader. Such a result follows much more easily from our approach.

A third application is to analyses for which it is not particularly appropriate to assume that exogenous traders' orders are normally distributed. Gorton and Pennacchi (1993) assume that exogenous traders' orders are binomially distributed and that the terminal value of the asset is normally distributed. Rather than solve for a linear equilibrium, they restrict market makers to choose a price that is the best linear unbiased estimate of the terminal value of the asset. Our results indicate that this combination of distributions precludes linear equilibria. However, if it were sensible to assume that the terminal value of the asset was binomially distributed, then a linear equilibrium can be computed and is described in Table 3.1, Part B.

3.2. A Variable Number of Symmetrically Informed Strategic Traders

The previous analysis took the number of strategic traders as given (and equal to the number of traders with private information), making it useful if one wishes to understand the effects of competition among strategic traders or the effect of regulations that alter the number of such traders. It is not appropriate if one wishes to consider traders' decisions to acquire or trade on private information. Such analyses provide insight into how much and how quickly information is impounded in prices and issues surrounding the regulation of insider trading (the desire to obtain information and the willingness to trade on one's private information).[30] For these situations, one must be able to compute a linear equilibrium for all N.

For the information acquisition problem, traders make some decisions prior to acquiring private information. If one studies who trades on their private information, all decisions may be made after obtaining the information. Whether all decisions are made

[30] See Kyle (1986a), Admati and Pfleiderer (1988), Khanna et al. (1994), Fishman and Hagerty (1992), Holden and Subrahmanyam (1992), Leland (1992), Foster and Viswanathan (1993), Brown and Zhang (1992) for examples of such applications.

prior to obtaining private information is important because it determines whether second moments must exist. If no decisions are made prior to learning the private information, second moments need not exist. If not, then second moments must exist. To see this, notice that, using (3.2), expected profits of acquiring information when $N-1$ others also acquire it are

$$\mathrm{E}\big[\mathrm{E}[(v-\mu_N-\lambda(X+u))x \mid s]\big] = \frac{1}{(N+1)\lambda_N}\mathrm{E}[(s-\mu_N-\lambda_N \mathrm{E}[u])^2].$$

Clearly, this is well defined only when s has a finite second moment.

If traders make some decisions prior to learning the private information, necessary and sufficient conditions for a linear equilibrium for all N will include a requirement that second moments exist. So, first we will determine necessary and sufficient conditions for a linear equilibrium for all numbers of strategic traders and then add the condition that the distributions have finite second moments. By Theorem 3.2 there is a linear equilibrium for a fixed N if and only if $\hat{s}$ has the same distribution as $(N+1)\lambda_N$ times the N-fold average of $\hat{u}$. It will be convenient to rewrite this. Let $w_i =^d \hat{u}$ for $i = 1, 2, \ldots, N$ and let $\omega_N \equiv \sum_{i=1}^N w_i$. In other words, ω_N is the sum of N i.i.d. random variables each of which has the same distribution as $\hat{u}$. Given this, the necessary and sufficient conditions presented in Theorem 3.2 can be rewritten as

$$\left(\frac{N}{(N+1)\lambda_N}\right)\hat{s} =^d \omega_N, \tag{3.4}$$

where $=^d$ denotes that two random variables are identical in distribution.

In this form, our task is to find distributions for s and u such that, for all N, the former is a normalized sum of i.i.d random variables all with the same distribution as u. One version of the Central Limit Theorem tells us that the class of limiting distributions of normed sums of i.i.d. random variables is the class of stable distributions.[31] Since the distribution of $\hat{s}$ is exogenous, it must be independent of N. Hence $\hat{s}$ and $\hat{u}$ must be stable random variables. There are many equivalent ways to describe this class. The following is a standard definition.

Definition. The distribution function F is stable if, for every $a_1, a_2 > 0$ and b_1, b_2, there are constants $a > 0$ and b such that $F(a_1 y + b_1) * F(a_2 y + b_2) = F(ay + b)$. Further, a random variable is stable if its distribution function is.[32]

To connect this to our analysis, notice that if y, y_1, and y_2 are each random variables with the same distribution function, then y is stable if $ay + b =^d y_1 + y_2$ for some constants $a > 0$ and b. This is readily generalized to the following: If $y, y_1, y_2, \ldots, y_n$ are each random variables with the same distribution, then y is stable if, for each n, there are constants a_n, b_n such that $a_n y + b_n =^d \sum_{i=1}^n y_i$.[33] This immediately implies that (3.4) is met for all N if and only if $\hat{s}$ and $\hat{u}$ are stable. Finally, since linear transformations of stable random variables are stable, s and u must be stable random variables too.

Theorem 3.4. *If privately informed traders make all decisions after learning the information, then there is a linear equilibrium for all N if and only if s and u are stable random variables with finite expectations. Further, for each N, there is only one linear equilibrium.*

[31] For details, see Gnedenko and Kolmogorov (1968) or Loève (1960).

[32] For two distributions F and G, $F * G$ is the convolution.

[33] Note that $b_n = 0$ if the means of the random variables are zero and y is strictly stable.

Since stable distributions or laws play a prominent role in the analysis, some additional information will be useful. Unfortunately, except in rare circumstances (see below), stable laws do not have explicit expressions for either their densities or distributions. As a result, characteristic functions are used to describe them and to derive most of their properties. There are at least three standard representation theorems for stable laws. We provide one based on Zolotarev (1986).

REPRESENTATION THEOREM. *The characteristic function ϕ of any nondegenerate stable distribution can be written as*

$$\log \phi(t) \equiv \delta(it\gamma - |t|^{\alpha} + it\omega_A(t, \alpha, \eta)),$$

where

$$\omega_A(t, \alpha, \eta) = \begin{cases} |t|^{\alpha-1}\eta \tan(\pi\alpha/2) & \text{if } \alpha \neq 1 \\ -\eta(2/\pi)\ln|t| & \text{if } \alpha = 1 \end{cases}$$

for $0 < \alpha \leq 2, -1 \leq \eta \leq 1, \delta > 0$ *and* γ *a finite, real number.*

Clearly, every stable law can be completely described by four parameters: α, η, γ, and δ. For convenience, if y is stable, we will write it as $Y(\alpha, \eta, \gamma, \delta)$. There are a number of properties of stable distributions that are useful. First, if $Y(\alpha, \eta, \gamma, \delta)$ is stable, then there exist real parameters a, b so that $Y(\alpha, \eta, \gamma, \delta) =^d aY(\alpha, \eta, \gamma', \delta') + \delta' b$. Hence, γ and δ are location parameters and it is common to use standard stable distributions, $\gamma = 0, \delta = 1$.[34] Second, there is a version of symmetry as follows: If $Y(\alpha, \eta, \gamma, \delta)$ is stable then $-Y(\alpha, \eta, \gamma, \delta) =^d Y(\alpha, -\eta, -\gamma, \delta)$. Third, any stable random variable can be written as a linear combination of independent stable random variables and any linear combination of stable random variables is stable. The last useful property is that stable laws are unimodal and that α (sometimes referred to as the stable exponent) describes how "peaked" the stable distribution is.

We mentioned above that there are a few stable laws that can be readily described by their densities. These include the normal distribution ($\alpha = 2$, any η), the Cauchy distribution ($\alpha = 1, \eta = 0$) for which no moments exist and whose density is

$$g(y; 1, 0) = \frac{1}{2}\left(\frac{\pi^2}{4} + y^2\right)^{-1},$$

and the Lèvy ($\alpha = 1/2, \eta = 1$) whose density is

$$g\left(y; \frac{1}{2}, 1\right) = \begin{cases} \frac{y^{-3/2}e^{-1/4y}}{2\sqrt{\pi}} & y > 0 \\ 0 & y \leq 0. \end{cases}$$

From the symmetry property, the symmetric reflection of the Lèvy distribution, $g(y, \frac{1}{2}, -1)$, is also stable. Also, from the third property, linear combinations of these yield stable distributions too.

Finally, we can use the third property to extend our characterization theorem when the exogenous orders are i.i.d. If they are and they are stable, then their sum is stable and we will have a linear equilibrium for all N if and only if s has that same distribution.

[34] In particular, if the mean of a stable random variable is zero, $\gamma = 0$.

As we explained above, if strategic traders make decisions before learning their private information, an additional condition is needed: finite second moments. For the stable distributions, if $\alpha = 2$, the stable law is the normal distribution. If $1 < \alpha < 2$, stable laws have finite means but infinite variances, and for $\alpha \leq 1$, stable laws have neither finite means nor variances.[35] Hence, there is only *one* member of the stable laws that has finite second moments: normal distributions.[36] This is important if we wish to evaluate the endogenous entry of strategic traders in the trading game (e.g., if strategic traders have to pay a cost to acquire information).

THEOREM 3.5. *If privately informed traders make some decisions prior to learning the information, then there is a linear equilibrium for all N if and only if s and u are normal random variables and, for each N, there is only one linear equilibrium.*

Proof. The unconditional expected profit is defined only if the second moments of the distributions $\hat{s}$ and $\hat{u}$ are well defined. Thus the only stable distribution that is allowed is the normal distribution ($\alpha = 2$). Theorem 3.3 then ensures uniqueness of linear equilibrium for each N. □

Our conditions for a linear equilibrium for all N can be reexpressed as,

$$\log \phi_{\hat{s}}\left(\frac{Nt}{(N+1)\lambda_N}\right) = N \log \phi_{\hat{u}}(t) \qquad \forall t.$$

Two complex numbers are equal if their real and imaginary parts are equal. If $\hat{s}$ and $\hat{u}$ are stable, then using the Representation Theorem and setting the real parts equal, we obtain

$$\delta\left(\frac{N}{(N+1)\lambda_N}\right)^{\alpha} |t|^{\alpha} = N\delta' |t|^{\alpha'} \qquad \forall t,$$

where primes denote parameters of the characteristic function of $\hat{u}$. Since this must hold for all t, $\alpha = \alpha'$; that is, $\hat{s}$ and $\hat{u}$ have the same stable exponent. Further, λ_N is

$$\lambda_N = \left(\frac{\delta}{N\delta'}\right)^{1/\alpha} \left(\frac{N}{N+1}\right).$$

By setting the imaginary parts equal and using the fact that the real parts are equal, one shows that $\eta = \eta'$. To complete the analysis we observe that the derivative with respect to t of a random variable's characteristic function evaluated at $t = 0$ is the first moment and we have assumed that the first moments exist. Since both $\hat{s}$ and $\hat{u}$ are nondegenerate mean zero random variables, $\gamma = 0$ and $\gamma' = 0$. Thus the other condition that comes from equating the imaginary parts holds automatically and we see that we have no restrictions on the relationship between δ and δ'. This discussion thus yields Theorem 3.6.

[35] See Gnedenko and Kolmogorov (1968, chap. 7).

[36] Noldeke and Troger (1998) obtain a stronger version of this theorem. In particular, they show that requiring a linear equilibrium for N_1 and N_2 (as opposed to all N) leads to normality when the second moments are finite.

THEOREM 3.6. *There is a linear equilibrium for all N iff $\hat{s}$ and $\hat{u}$ are both stable random variables with the same α's, η's, and γ's but whose δ's may differ. For any such pair of stable random variables, the slope of the only equilibrium linear price function is*

$$\lambda_N = \left(\frac{\delta}{N\delta'}\right)^{1/\alpha}\left(\frac{N}{N+1}\right).$$

Since the normal distribution has δ equal to the variance and $\alpha = 2$, Theorem 3.6 yields the same answer as the calculation of λ_N in Theorem 3.3 when second moments exist (as they do for the normal distribution).

3.3. Implications of Linear Equilibria in Kyle Type Models with Symmetric Information

As mentioned above, when strategic traders are symmetrically informed and distributions satisfy the necessary and sufficient conditions for a linear equilibrium, then there is only one. Further, we have characterized the intercept and slope for a large number of examples—those described in Table 3.1 as well as the class of stable distributions. There are a couple of features that are common to all of these examples.

PROPOSITION 3.1. *If there is a linear equilibrium and both s and u have finite second moments or if they are stable random variables with finite first moments, then λ_N declines in N.*

Proof. Simply examine the expressions for the slope in Theorems 3.3 and 3.6 □

Our final point relies on Theorems 3.5 and 3.6. They show that there is a linear equilibrium in Kyle type models when some traders either choose to become informed or choose whether to trade on their information if and only if the distributions are stable. An important property of stable random variables is that their support is the whole real line. No random variable on a bounded subset of the real line can be stable. Consequently, we know that the distributional conditions derived in Theorems 3.5 and 3.6 imply that there is always a positive probability that the price of the asset is negative in a linear equilibrium.

3.4. Differentially Informed Strategic Traders

In this subsection, we consider a situation which is more commonly studied in the rational expectations literature—strategic traders each see their own private signal about the terminal value of the asset. As is standard, we assume that the ith strategic trader's signal is $s_i = v + \epsilon_i$ where the ϵ_i's are mean zero, i.i.d. random variables. The market makers' problem is unaffected by this change and so we seek conditions such that $\mathrm{E}[v \mid z]$ is a linear function of z. The most important difference between this and the previous formulation is that strategic traders are not symmetrically informed. As a result, they now play a game of incomplete information.

The ith strategic trader chooses an order, x_i, to maximize her expected wealth given her signal, s_i. Since $z = x_i + X_{\sim i} + u$ and the price function is linear ($\mu_N + \lambda_N z$ for $\lambda_N > 0$), she solves

$$\max_{x_i} \mathrm{E}[x_i(v - \mu_N - \lambda_N(x_i + X_{\sim i} + u)) \mid s_i].$$

The first-order condition is

$$x_i = \left(\frac{1}{2\lambda_N}\right)\left(\mathrm{E}[v \mid s_i] - \mu_N - \lambda_N \mathrm{E}[X_{\sim i} \mid s_i]\right) \tag{3.5}$$

and the second-order condition is satisfied if $\lambda_N > 0$. To complete the analysis, assume that $\mathrm{E}[\hat{v} \mid \hat{s}_i] = \xi \hat{s}_i$ for $i = 1, 2, \ldots, N$.[37] Finally, in contrast to prior sections, we focus on equilibria in which strategic traders choose strategies that are linear in their signal, $x_j(\hat{s}_j) = a_j + b_j \hat{s}_j$.[38]

PROPOSITION 3.2. *Given that market makers choose a linear price function, a Bayes–Nash equilibrium has strategic traders choosing*

$$x(\hat{s}_i) = \left(\frac{\xi}{\lambda_N(2 + \xi(N-1))}\right)\hat{s}_i \equiv \beta_N \hat{s}_i.$$

To find sufficient conditions for a linear equilibrium, we begin with a lemma that provides necessary and sufficient conditions on the distributions of v and ϵ_i so that $\mathrm{E}[\hat{v} \mid \hat{s}_i] = \xi \hat{s}_i$. Let ϕ_ϵ be the common characteristic function for the ε_i's.

LEMMA 3.1. $\mathrm{E}[\hat{v} \mid \hat{s}_i] = \xi \hat{s}_i$ *if and only if* $\phi_{\hat{v}}(t) = \phi_\epsilon(t)^{\xi/(1-\xi)}$.

As an aside, many models use the "true value plus noise" type signal: $s_i = v + \epsilon_i$ with mean zero, i.i.d. ϵ_i's. Authors[39] often assume that $\mathrm{E}[v \mid s_i]$ is linear in s_i and rely on Ericson (1969) to write $\mathrm{E}[v \mid s_i] = (1-\tau)\mathrm{E}[v] + \tau s_i$ for $\tau = \mathrm{Var}[v]/(\mathrm{Var}[v] + \mathrm{Var}[\varepsilon_i])$.[40] Since $\mathrm{E}[\hat{v} \mid \hat{s}_i] = \xi \hat{s}_i$ implies that $\mathrm{E}[v \mid s_i] = (1-\xi)\mathrm{E}[v] + \xi s_i$, Lemma 3.1 provides a characterization of the class of distributions for which that conditional expectation *is* linear and a (slight) generalization of Ericson's characterization of the conditional expectation. If second moments are finite, $\xi = \tau$ and if not, ξ can still be determined from the characteristic functions in the same way that we characterized the slope of our price function in Section 3.3.[41]

Returning to our problem, we can use Lemma 3.1 and apply Ferguson's Lemma to obtain conditions on the distributions of v and u that support a linear equilibrium, where by this we now mean a sequentially rational Bayes–Nash equilibrium in which market makers choose a linear pricing rule and strategic traders' orders are linear functions of their signals.

THEOREM 3.7. *There is a linear equilibrium if* $(\vec{s}, u)$ *have finite expectations and*

$$\phi_{\hat{v}}(N\beta_N t)^{\frac{2-\xi}{N\xi}} = \phi_{\hat{v}}(\beta_N t)^{N(1-\xi)/\xi}\phi_{\hat{u}}(t) \qquad \forall t. \tag{3.6}$$

[37] Since $s_i = v + \epsilon_i$, $\mathrm{E}[s_i] = \mathrm{E}[v]$. Substituting, $\mathrm{E}[\hat{v} \mid \hat{s}_i] = \xi \hat{s}_i$ is equivalent to $\mathrm{E}[v \mid s_i] = (1-\xi)\mathrm{E}[v] + \xi s_i \equiv \theta + \xi s_i$.

[38] Recall that when strategic traders were symmetrically informed, we showed that they would employ the same trading strategy in equilibrium and that it would be linear in their common signal. To show this, we only required that the equilibrium price be a linear function of the aggregate order flow. Unfortunately, when strategic traders are differentially informed, we must restrict attention to equilibria in which it is optimal for strategic traders to use linear trading strategies. It remains an open question whether there are linear equilibria when strategic traders employ nonlinear trading strategies. Consequently, in this subsection, we will only be able to derive sufficient conditions for a linear price function.

[39] See, for example, Vives (1989) or Hwang (1993) and the references therein.

[40] Obviously, Ericson's proof assumes that the distributions have finite second moments.

[41] One could also use the condition on the characteristic functions in the lemma to construct a table analogous to Tables 3.1 and 4.1 (the latter appears in Section 4) and describe some of the members of the class of distributions.

For fixed N, (3.6) characterizes the class of distributions that support a linear equilibrium in Kyle type models with differentially informed strategic traders. Unfortunately, there is no nice interpretation unless the powers in (3.6) are integers.[42] If so, (3.6) requires that $\hat{u}$ plus the sum of $N(1-\xi)/\xi$ random variables with the same distribution as $\beta\hat{v}$ have the same distribution as the sum of $2-\xi/N\xi$ random variables with the same distribution as $N\beta\hat{v}$.

Extending our analysis to characterize distributions that support a linear equilibrium for all N is difficult. If the powers in (3.6) are not integers, there is no nice interpretation at all. If they are integers, $\hat{v}$ can be interpreted as the sum of random variables but not as the sum of N i.i.d. random variables. Instead, it is the sum of independent but not identically distributed random variables. Thus, for there to be a linear equilibrium for all N when the powers in (3.6) are integers, then v must be an element of the class of limiting distributions of normed sums of independent but not identically distributed random variables—the infinitely divisible laws (a class which contains the stable laws).[43] Little more can be said because there is no simple representation theorem for the characteristic function of an infinitely divisible random variable.

3.4.1. *Examples and Implications.* We cannot construct tables of examples as we did for the symmetric information version. However, we can compute examples when v and u are stable by using the Representation Theorem. Substituting into (3.6) and setting the real parts equal (suppressing the subscript on β_N),

$$\delta\left(\frac{2-\xi}{N}\right)\left(\mid N\beta t\mid^{\alpha}\right) = N(1-\xi)\delta \mid \beta t\mid^{\alpha} + \xi\delta' \mid t \mid^{\alpha'} \qquad \forall t,$$

where primes denote the parameters of the characteristic function of $\hat{u}$. Thus, $\alpha = \alpha'$ and

$$\beta^{\alpha} = \left(\frac{N\xi\delta'}{\delta\big((2-\xi)N^{\alpha} - N^2(1-\xi)\big)}\right).$$

Using the definition of β,

$$\lambda_N = \left(\frac{\xi}{2+\xi(N-1)}\right)\left(\frac{\delta\big((2-\xi)N^{\alpha} - N^2(1-\xi)\big)}{N\xi\delta'}\right)^{1/\alpha}. \tag{3.7}$$

As both $\hat{v}$ and $\hat{u}$ are strictly stable ($\gamma = \gamma' = 0$), setting the imaginary parts equal yields

$$\frac{\eta'}{\eta} = \frac{\beta\delta(2-\xi-N(1-\xi))}{\xi\delta'}.$$

Thus, (3.7) characterizes the slope of the equilibrium linear price function when v and u are stable.[44] As before, the intercept is simply $\mathrm{E}[v] - \lambda_N \mathrm{E}[u]$. For a linear equilibrium,

[42] We can provide sufficient conditions for this. Let c_1, c_2, and c_3 be integers and note that if $N(1-\xi) = c_1\xi$ and $(2-\xi) = c_2N\xi$, both powers are integers. Substituting the latter into the former, $N^2c_2\xi - N = c_1\xi$. If we assume that $c_1 = c_3N$ then $\xi = 1/(c_2N - c_3)$. Thus, we can choose c_2 and c_3 as functions of N to ensure that ξ is a positive fraction and make the powers in (3.6) integers.

[43] As we mentioned previously, the class of stable distributions is defined as the set of distributions for random variables that are the limit of normed sums of independent and identically distributed random variables as the number of random variables gets large. The class of infinitely divisible distributions is larger (and contains the class of stable distributions) because it is defined as the set of distributions for random variables that are the limit of normed sums of independent random variables (which need not be identically distributed). See Gnedenko and Kolmogorov (1968) or Zolotarev (1986) for additional details.

[44] Notice that Admati and Pfleiderer's (1988) results for the normal case are obtained by setting $\alpha = 2$ in (3.7).

the slope must be positive. Examining (3.7), one notes that if $\alpha < 2$, then for large values of N, λ_N is negative. In other words, if the random variables are stable, there is an upper bound on the number of strategic traders that admit a linear equilibrium. The only exception is the normal where λ_N is well defined for all N.

As we mentioned above, if the powers in (3.6) are integers, the class of limiting distributions is not the stable laws but the infinitely divisible laws. As a result, we cannot use the above facts for the stable distributions to determine sufficient conditions for a linear equilibrium for all N. However, in the Appendix, we prove the following theorem.

THEOREM 3.8. *If differentially informed strategic traders use linear trading strategies and v, u have finite second moments, then there is a linear equilibrium for all N iff v and u are normal.*

We close this subsection with some comparative statics results when v and u are stable because they yield some surprises. Define $N^*(\alpha, \xi)$ so that if there are more than $N^*(\alpha, \xi)$ strategic traders, then $\lambda_N < 0$ and so there is no linear equilibrium.[45] The first surprise is that the aggressiveness of the strategic traders (β) is not monotone in the number of strategic traders unless v and u are normally distributed. To see this, differentiate the expression for β above to obtain

$$\frac{\partial \beta^\alpha}{\partial N} = -A\left(\frac{(2-\xi)(\alpha-1)N^\alpha - N^2(1-\xi)}{((2-\xi)N^\alpha - N^2(1-\xi))^2}\right) \qquad \forall N < N^*(\alpha, \xi),$$

for some positive constant A. If $\alpha = 2$ the derivative is always negative. Thus, the well-known result[46] for normally distributed random variables that each strategic trader trades less aggressively the more strategic traders there are holds with differentially informed strategic traders. However, if $\alpha \in (1, 2)$ (other stable distributions), the sign of the numerator is indeterminate—positive for small values of N but negative for relatively large values of N (but $N < N^*(\alpha, \xi)$). We summarize all of this discussion in Proposition 3.3.

PROPOSITION 3.3. *If there are $N < N^*(\alpha, \xi)$ differentially informed strategic traders and the random variables are stable, then* (i) *if $\alpha = 2$ (the random variables are normal), each trader trades less aggressively the more of them there are* $((\partial\beta/\partial N) < 0)$ *and* (ii) *if $\alpha \in (1, 2)$, then for small N an increase in the number of strategic traders leads them to trade less aggressively but for larger N an increase leads them to trade more aggressively. However, the slope of the equilibrium linear price function is strictly decreasing in the number of strategic traders for all $\alpha \in (1, 2]$.*

Thus, even though strategic traders' aggressiveness first decreases and then increases unlike the usual case, the slope of the price function always declines in the number of strategic traders.

4. GLOSTEN–MILGROM, EASLEY–O'HARA TYPE MODELS

The important difference between GMEO and Kyle type models is that the former assumes that exactly one order gets executed and that this order is chosen randomly from the orders submitted by strategic and exogenous traders. Assume that a strategic

[45] $N^*(\alpha, \xi)$ is the largest integer smaller than that value of N that makes the right-hand side of (3.7) negative.
[46] See Holden and Subrahmanyam (1992) or Foster and Viswanathan (1993).

trader's order is executed with probability ρ and, conditional on a strategic trader's order being selected, each of their orders is equally likely to be selected. We will initially consider the case when strategic traders are differentially informed and then specialize our results to the case when they are symmetrically informed.[47] Recall that $s_i = v + \epsilon_i$ with i.i.d. ϵ_i's, and $\mathrm{E}[v \mid s_i] = \theta + \xi s_i$ (equivalently, $\mathrm{E}[\hat{v} \mid \hat{s}_i] = \xi \hat{s}_i$). We simplify by only considering the case when the exogenous trades are i.i.d., $u_i =^d u_j \equiv^d u^*$. One final observation will be of use. Since $s_i = v + \epsilon_i$ with i.i.d. ϵ_i's, then $s_i =^d s_j$ and we will refer to this common distribution as the distribution of s.[48]

In the second stage, market makers choose prices after observing the randomly chosen order that will be executed, z. Although all market makers submit prices, only one will execute the order (the order cannot be divided). As before, the solution is to set $p = \mathrm{E}[v \mid z]$. A linear equilibrium arises if this is a linear function of the order, $p_G(z) = \mu_G + \lambda_G z$ with $\lambda_G > 0$.

Given this, each strategic trader solves[49]

$$\max_{x_i} \mathrm{E}[v - \mu - \lambda z \mid s_i] x_i.$$

Whether the strategic trader is certain that her order will be executed or not is unimportant because she is risk neutral. In either case, the only payoff relevant event is when $z = x_i$. Using this, the first-order condition is

$$x_i(\hat{s}_i) = \frac{\xi}{2\lambda} \hat{s}_i \equiv \beta \hat{s}_i,$$

and the second-order condition is satisfied because $\lambda > 0$.

THEOREM 4.1. *In GMEO type models, there is a linear equilibrium if and only if the probability that a strategic trader's order is executed is* $1/2$, $(\vec{s}, u^*)$ *have finite expectations, and* $(\xi/2\lambda)\hat{s} =^d \hat{u}^*$. *Further, if there is a linear equilibrium, then there is only one. The intercept is* $\mathrm{E}[v]$ *and the slope is characterized by the condition on the distributions of* s *and* u^*. *Finally, the conditions are the same when we seek a linear equilibrium for all* N.

Another representation of the condition on the distributions is $x_i =^d u^*$, which is very similar to Theorem 3.1. The symmetric information version is obtained when every strategic trader observes s and $\mathrm{E}[v \mid s] = s$. The solution is found by setting $\theta = 0$ and $\xi = 1$. There are a couple of odd features about our theorem. It requires that the probability that the order executed comes from a strategic trader be $1/2$, independent of the number of exogenous and strategic traders there are. In many situations, it is more natural to assume that this probability is equal to the fraction of strategic traders. In such situations, our theorem implies that there will not, in general, be a linear equilibrium. Looked at another way, if there is to be a linear equilibrium, regardless of the number of strategic traders, $\rho = 1/2$ and so the equilibrium price function does not depend on the number (or fraction) of strategic traders.

[47] We followed a different approach for Kyle type models because a strategic trader's problem is very different in the two cases. In GMEO type models, only one order is executed and so a strategic trader's problem is the same in the two cases.

[48] In other words, each signal is drawn from the same distribution, the distribution of s.

[49] Again, we suppress the regime identifiers when no confusion will arise.

4.1. Examples and Comparisons with Kyle Type Models

As we did for Kyle type models with symmetric information, we can use Theorem 4.1 to create Table 4.1 which characterizes linear equilibria for different combinations of distributional assumptions. This is done by noting that we can express the condition that $(\xi/2\lambda)\hat{s} =^d \hat{u}^*$ as

$$\phi_s(\beta t) = e^{it(\beta \mathrm{E}[s] - \mathrm{E}[u^*])}\phi_{u^*}(t) \qquad \forall t.$$

Using the appropriate characteristic functions yields the solutions reported in Table 4.1. At first glance, it may appear that GMEO and Kyle type models return the same solutions. This is only true when there is one exogenous trader in Kyle type models. The reason is that u in Table 3.1 is the sum of the L exogenous traders' orders whereas u^* is *one* exogenous trader's order. Since the second moment of the sum of their orders is strictly larger than the second moment of any exogenous trader's order (equal if the exogenous traders' orders are perfectly correlated) and since the slope of the equilibrium price function is the covariance of the terminal value of the asset and the order divided by the variance of the order, we have:

PROPOSITION 4.1. *If there are at least two exogenous trades that do not perfectly covary, then the slope of the linear equilibrium price function in GMEO type is greater than in Kyle type models.*

TABLE 4.1
GMEO Type Models

Distribution	Distributions of u and s	Linear equilibrium price function
Normal	$u \sim \mathcal{N}(\alpha, \theta), \quad s \sim \mathcal{N}(\delta, \gamma)$	$p(z) = \delta + \left(\frac{\xi}{2}\sqrt{\frac{\gamma}{\theta}}\right)(z-\alpha)$
Laplace	$u \sim LP(\alpha, \theta), \quad s \sim LP(\delta, \gamma)$	$p(z) = \delta + \left(\frac{\xi\gamma}{2\theta}\right)(z-\alpha)$
Logistic	$u \sim LOG(\alpha, \theta), \quad s \sim LOG(\delta, \gamma)$	$p(z) = \delta + \left(\frac{\xi\gamma}{2\theta}\right)(z-\alpha)$
Extreme value	$u \sim EV(\alpha, \theta), \quad s \sim EV(\delta, \gamma)$	$p(z) = \delta + \frac{k\gamma}{2} + \left(\frac{\xi\gamma}{2\theta}\right)(z-\alpha)$
Gamma	$u \sim \mathcal{G}(\alpha, \theta), \quad s \sim \mathcal{G}(N\alpha, \gamma)$	$p(z) = \alpha + \left(\frac{\xi\gamma}{2\theta}\right)(z-\alpha)$
Chi–square	$u \sim \chi^2(\alpha), \quad s \sim \chi^2(\alpha)$	$p(z) = \alpha + \frac{\xi}{2}(z-\alpha)$
Inverse Gaussian	$u, \quad s \sim \mathcal{IG}(\alpha, \theta)$	$p(z) = \alpha + \frac{\xi}{2}(z-\alpha)$
Pearson Type III	$u \sim P3(\alpha, \theta, \nu), \quad s \sim P3(\delta, \gamma, \nu)$	$p(z) = \delta + \gamma\nu + \left(\frac{\xi\gamma}{2\theta}\right)(z-\alpha-\theta\nu)$
Uniform	$u \sim \mathcal{U}(\alpha, \theta), \quad s \sim \mathcal{U}(\delta, \gamma)$	$p(z) = \frac{\gamma+\delta}{2} + \left(\frac{\xi(\gamma-\delta)}{2(\theta-\alpha)}\right)(z - \frac{\theta-\alpha}{2})$
Triangular	$u \sim \mathcal{T}(\alpha, \theta), \quad s \sim \mathcal{T}(\alpha, \gamma)$	$p(z) = \gamma + \left(\frac{\xi}{2}\right)(z-\theta)$
Binomial	$u, \tau \sim \mathcal{B}(\alpha, \theta), \quad s = k\tau$	$p(z) = k\alpha\theta + \left(\frac{k\xi}{2}\right)(z-\alpha\theta)$
Negative Binomial	$u, \tau \sim \mathcal{NB}(\alpha, \theta), \quad s = k\tau$	$p(z) = \left(\frac{k\alpha(1-\theta)}{\theta}\right) + \left(\frac{k\xi}{2}\right)\left(z - \frac{\alpha(1-\theta)}{\theta}\right)$
Pascal	$u, \tau \sim \mathcal{P}(\alpha, \theta), \quad s = k\tau$	$p(z) = \frac{k\alpha}{\theta} + \left(\frac{k\xi}{2}\right)\left(z - \frac{\alpha}{\theta}\right)$
Poisson	$u, \tau \sim P(\alpha), \quad s = k\tau$	$p(z) = \alpha + \left(\frac{k\xi}{2}\right)(z-\alpha)$
Geometric	$u, \tau \sim G(\alpha), \quad s = k\tau$	$p(z) = \frac{1}{\alpha} + \left(\frac{k\xi}{2}\right)\left(z + \frac{1}{\alpha}\right)$

5. INDIVIDUAL ORDER TYPE MODELS

Individual order type models are similar to Kyle type models in that every strategic and exogenous trader's order is executed. The key difference between them is that in IO type models, market makers see the whole vector of orders, not just the order imbalance (aggregate net order flow). We assume that the position of the order in the order vector conveys no information to market makers[50] and model this by assuming that the order vector market makers see, Z, is a permutation of the vector $(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_L)$ where n can be larger than or equal to N, the number of strategic traders. Whether n is strictly greater than N depends on whether strategic traders are permitted to submit multiple orders, an issue that will affect the subsequent analysis.

In an IO model, a linear equilibrium has $p(Z) = \mu + \sum_{i=1}^{n+L} \lambda^i z_i$, where $Z = (z_1, z_2, \ldots, z_{n+L})$. Since position is uninformative, for any permutation of Z all orders must be executed at the same price. This implies that $\lambda^i = \lambda^j$ for all i, j. Thus, in any *linear* equilibrium, $p(Z) = \mu + \lambda \sum_{i=1}^{n+L} z_i \equiv \mu + \lambda z$. Since market makers are risk neutral, $\mathrm{E}[v \mid Z] = p(Z)$ in equilibrium. Noting that in Kyle type models, $\mathrm{E}[v \mid z]$ is a linear function of z, the order imbalance, we have the following theorem.

THEOREM 5.1. *Any distributions that support a linear equilibrium in an individual order model support a linear equilibrium in Kyle type models and have* $\mathrm{E}[v \mid Z] = \mathrm{E}[v \mid z]$.

This last condition explains why not every linear equilibrium in a Kyle type model is a linear equilibrium in an IO model. Theorem 5.1 tells us that distributions that support a linear equilibrium in an IO model are a subset of the distributions that support a linear equilibrium in a Kyle type model, the subset being defined by the added condition that $\mathrm{E}[v \mid Z] = \mathrm{E}[v \mid z]$. Thus, we must study the effect of this last condition on the results obtained in Section 3.

5.1. An Exogenous Number of Symmetrically Informed Strategic Traders

In this subsection, we maintain the assumptions used in Section 3.1 and proceed by assuming that strategic traders are only allowed to submit one order. (We relax this restriction below.) As a result, a strategic trader's problem is the same as in Section 3.1 and so (3.1) characterizes her best reply and (3.2) characterizes the unique Nash equilibrium given $p(Z) = \mu_N + \lambda_N z$.

THEOREM 5.2. *If strategic traders submit one order each, then there is a linear equilibrium in an IO model iff there is one strategic and one exogenous trader, their orders have the same distribution, and (s, u_1) have finite expectations. These latter conditions are equivalent to $\hat{s} =^d 2\lambda \hat{u}_1$.*

The intuition relies on the fact that market makers observe each order. Since they know that there are N strategic traders and each submits exactly one order, market makers realize that N of the $N + L$ elements of the order vector were submitted by strategic traders. Since symmetrically informed strategic traders submit the same order, the conditional probability that any N elements of Z are this common order is zero if the N selected elements are not all the same. As a result, $\mathrm{E}[v \mid Z]$ is a weighted average of $\mathrm{E}[v]$ and $\mathrm{E}[v \mid \zeta] = \mu_N + 2\lambda_N \zeta$ for ζ equal to the common value of the N selected

[50] We do this because this model is intended to capture salient features of call markets with market makers and it appears that market makers do not know who submits which orders in such markets.

elements and the weights are the conditional probabilities that the N selected elements were submitted by strategic traders given Z. But, by Theorem 5.1, $\mathrm{E}[v \mid Z] = \mathrm{E}[v \mid z]$ or $(N+1)\lambda_N$ times the expected order submitted by strategic traders conditional on Z must equal z, the order imbalance. This is a very stringent condition and can be met only if $N = L = 1$. The conditions on the distributions of x_1 and $\hat{u}_1$ (or equivalently $\hat{s}$ and $\hat{u}_1$) follow from Theorems 3.1 and 3.2.

The proof of Theorem 5.2 relies heavily on the assumption that each strategic trader is allowed to submit exactly one order. There are circumstances where such a condition is sensible, as, for example, if market makers can commit to not executing an order vector that contains more than one order for each trader. Generally, though, we expect that strategic traders cannot be restricted to submitting only one order each and so we turn to conditions under which there is a linear equilibrium in the IO model when they can split their order.

In what follows, we restrict attention to environments with exactly one strategic trader.[51] Given this, conditional on a price function, the manner in which the order is split is payoff irrelevant to the strategic trader. Consequently, (3.1) continues to define her best reply and, given that there is only one strategic trader, her equilibrium choice too. Also, with one strategic trader, there is no distinction between symmetrically and differentially informed strategic traders. Even though how the strategic trader's total order, x_1, is split is payoff irrelevant, it will affect market makers' inferences. Let a split be a vector $R = (r_1, r_2, \ldots, r_k)$ with $r = \sum_{i=1}^{k} r_i$ such that $x_1 = r$. As before, there are $L \geq 1$ exogenous traders submitting orders $(u_1, u_2, \ldots, u_L)$ and $u = \sum_{i=1}^{L} u_i$.

By Theorem 5.1, we know that conditions for a linear equilibrium in an IO model are those from Theorems 3.1 and 3.2 (equivalently Theorem 3.7 when $N = 1$) plus the requirement that $\mathrm{E}[v \mid Z] = \mathrm{E}[v \mid z]$. The former conditions require that x_1 be equal in distribution to $\hat{u}$. This is equivalent (by Theorem 3.2) to requiring that the signal less its expected value, $\hat{s}$, be equal in distribution to $2\lambda\hat{u}$. The final condition, $\mathrm{E}[v \mid Z] = \mathrm{E}[v \mid z]$, is trivially satisfied. Since we already require that $x_1 =^d \hat{u}$, the strategic trader can simply choose a split so that $r_i =^d \hat{u}_i$ for $i = 1, 2, \ldots, L$. If the strategic trader employs such a split, then the following theorem holds.

THEOREM 5.3. *If there are $L \geq 1$ exogenous traders and one strategic trader who may split her order as she chooses, then there is a linear equilibrium in an IO model under the* same *distributional assumptions that support a linear equilibrium in Kyle type models: $x =^d \hat{u}$ or $\hat{s} =^d 2\lambda\hat{u}$.*

Consequently, all of our results in Section 3 on the linear equilibrium in Kyle type models carry over to the linear equilibrium in an IO model and will not be repeated. We do note that this implies that the restriction to a linear equilibrium in an IO model is a very severe restriction. Observing the vector of submitted orders will generally result in a different equilibrium price function than observing only the order imbalance.

6. CONCLUSION

In a game-theoretic market microstructure model that has three standard market microstructure models as special cases, we sought necessary and sufficient conditions for linear

[51] Allowing strategic traders to split orders creates an immediate problem because a strategy now is not only how many shares each wishes to trade, but *how* to split the order. This latter dimension becomes so important that we leave considerations of conditions for a linear equilibrium in an IO model with two or more strategic traders for future work.

equilibria. In Kyle type models, market makers see the order imbalance, select prices, then execute orders at the price most favorable to the traders. With identically informed strategic traders, the necessary and sufficient condition for a *linear* equilibrium is that this common order must have the same distribution as an N-fold average of the exogenous traders' aggregate net order, $\hat{u}$. We provided a series of examples of combinations of distributions that produce a linear equilibrium and computed the equilibrium price function. We also showed that most of the results from the standard version (with normal distributions) are due to the focus on linear equilibria, not the normality assumptions.

Necessary and sufficient conditions for a linear equilibrium when the number of active strategic traders is endogenous are more stringent. If no decisions are made prior to obtaining private information, there is a linear equilibrium if and only if all random variables are stable (a class that includes the normal distribution). If some decisions are made prior to obtaining private information, there is a linear equilibrium if and only if the random variables are normally distributed.

We extended the analysis to differentially informed strategic traders using a standard "true value plus noise" signal. We provided a characterization theorem for the class of distributions that support a linear equilibrium and used it to derive linear equilibria when all random variables were stable. For this class, the slope of the price function was decreasing in the number of strategic traders. We showed that the results on existence of equilibrium and strategic traders' aggressiveness were sensitive to which stable distribution was chosen. If the random variables were normally distributed, there was a linear equilibrium for any number of strategic traders and their aggressiveness was increasing in their numbers. For all other stable random variables, there was a maximum number of strategic traders for which there was a linear equilibrium.

In GMEO type models, exactly one order is submitted, which market makers observe and price. It is executed at the price most favorable to the trader. We showed that necessary and sufficient conditions for a linear equilibrium are the same whether strategic traders are symmetrically or differentially informed, because when only one order is executed, whether a strategic trader knows what the other strategic traders know is unimportant. Necessary and sufficient conditions for a linear equilibrium are that each strategic trader's order must have the same distribution as each exogenous trader's order and the probability that the order executed comes from a strategic trader must be $\frac{1}{2}$. This latter condition is quite strong because it *requires* that the market mechanics ensure that the probability is $\frac{1}{2}$ regardless of the number or fraction of strategic traders. Finally, we noted that if there are exactly one strategic and one exogenous trader, then the linear equilibria in the GMEO and Kyle type models must be identical and supported by exactly the same distributions. If there is more than one trader of either type, the slope of the linear price function in GMEO type models is greater than for Kyle type models—the market is less deep in GMEO type models.

Individual order type models are designed to capture salient features of call markets with market makers. Thus, market makers observe the vector of orders (not just the order imbalance), cannot infer anything about who submitted the order from its position in the order vector, and choose prices. All orders are executed at the price most favorable to traders. We showed that the restriction to linear equilibrium in markets of this type is quite important; it implies that if there is a linear equilibrium in an IO type model then there is one in Kyle type models too (but not the converse).

We showed that if strategic traders cannot split their total order, then there is a linear equilibrium if and only if there is exactly one strategic and one exogenous trader and the distributions of their orders are the same. If strategic traders are allowed to split their total order, the analysis of how they do this in equilibrium is extremely complicated. However, when we restricted attention to one strategic trader who could split her order, we found that the conditions for a linear equilibrium are the same as in Kyle type models. As a result, all of our results about equilibria in Kyle type models carried over to this setting when there was one strategic trader.

We have thus presented a detailed analysis of the conditions that are required to support a linear equilibrium in models of market making. Our analysis focused not just on the Kyle model extensively analyzed in the literature but also the GMEO model and the individual order model. Most real world financial markets are closer to the GMEO and individual order models and thus our work represents a first step toward analyzing more realistic market structures.

APPENDIX

Proof of Theorem 3.1. Since the players' objective functions are not well defined unless the random variables have finite expectations, equilibrium exists only if they do. Given this, by Ferguson's Lemma, $\mathrm{E}[\hat{s} \mid \hat{z}] = \lambda_N \hat{z}$ iff

$$\frac{\partial}{\partial t_0}\phi_{\hat{s},\hat{z}}(t_0, t_1)\bigg|_{t_0=0} = \lambda_N \frac{\partial}{\partial t_1}\phi_{\hat{z}}(t_1).$$

Substituting using the definition of a characteristic function ($\phi_y(t) \equiv \mathrm{E}[e^{i(t\cdot y)}]$) and differentiating,

$$\mathrm{E}[i\hat{s}e^{it_1\hat{z}}] = \lambda_N \mathrm{E}[i\hat{z}e^{it_1\hat{z}}].$$

Note that $\hat{z} = \hat{u} + N\hat{x}$ and that $\hat{x}$ and $\hat{u}$ are independent because $\hat{s}$ and $\hat{u}$ are. Substituting and rearranging, we have

$$\mathrm{E}[i\hat{s}e^{it_1N\hat{x}}]\mathrm{E}[e^{it_1\hat{u}}] = \lambda_N\big(\mathrm{E}[iN\hat{x}e^{it_1N\hat{x}}]\mathrm{E}[e^{it_1\hat{u}}] + \mathrm{E}[e^{it_1N\hat{x}}]\mathrm{E}[i\hat{u}e^{it_1\hat{u}}]\big).$$

Using (3.2) to eliminate $\hat{s}$ and rearranging,

$$\mathrm{E}[e^{it_1\hat{u}}]\mathrm{E}[i\hat{x}e^{it_1N\hat{x}}] = \mathrm{E}[e^{it_1N\hat{x}}]\mathrm{E}[i\hat{u}e^{it_1\hat{u}}].$$

The final steps are simpler if we revert to characteristic function notation. Using primes to denote derivatives with respect to t_1,

$$\phi_{\hat{u}}(t_1)\phi'_{N\hat{x}}(t_1) = N\phi_{N\hat{x}}(t_1)\phi'_{\hat{u}}(t_1)$$

or

$$\phi_{\hat{u}}(t_1)\phi'_{N\hat{x}}(t_1) - N\phi_{N\hat{x}}(t_1)\phi'_{\hat{u}}(t_1) = 0.$$

If $\phi_{\hat{u}}(r_1) \neq 0$, then by continuity of the characteristic function there is an open neighborhood of r_1 where the function is nonzero. Hence, one can divide by $\phi_{\hat{u}}(t_1)$ in this neighborhood and obtain

$$\frac{\partial}{\partial t_1}\left(\frac{\phi_{N\hat{x}}(t_1)}{\phi_{\hat{u}}(t_1)^N}\right) = 0.$$

Since $\phi_{\hat{u}}(t_1)$ is not identically zero in this neighborhood of r_1, this means that

$$\phi_{N\hat{x}}(t_1) = C(r_1)\phi_{\hat{u}}(t_1)^N$$

for some complex constant $C(r_1)$.

Similarly, if $\phi_{N\hat{x}}(r_1) \neq 0$, a similar argument shows that

$$\frac{\partial}{\partial t_1}\left(\frac{\phi_{\hat{u}}(t_1)}{\phi_{N\hat{x}}(t_1)^{(1/N)}}\right) = 0.$$

As before, this leads to

$$\phi_{N\hat{x}}(t_1) = C(r_1)\phi_{\hat{u}}(t_1)^N$$

for some complex constant $C(r_1)$.

The final possibility is that $\phi_{\hat{u}}(r_1) = \phi_{N\hat{x}}(r_1) = 0$, in which case there is nothing to prove.

Hence, we have a relationship that we desire between the characteristic functions, except that we need to show that the constant $C(r_1)$ is independent of r_1. If that were the case, since all characteristic functions evaluated at 0 equal 1, we have $C(r_1) = C = 1$.

To show that the constant $C(r_1)$ is independent of r_1, we need to impose our technical condition. Otherwise the theorem does not hold. For example, let N=1 and let each t_1

$$\text{Int}_1(t_1) = \text{largest integer smaller than } t_1$$

and

$$\text{Int}_2(t_1) = \text{smallest integer larger than } t_1.$$

Now define

$$\phi_{\hat{u}}(t_1) = \begin{cases} 1-(t_1 - \text{Int}_1(t_1)) & \text{if Int}(t_1) \text{ is even} \\ 1-(\text{Int}_2(t_1) - t_1) & \text{if Int}(t_1) \text{ is odd} \end{cases}$$

and define

$$\phi_{\hat{x}}(t_1) = 2[\phi_{\hat{u}}(t_1) - \tfrac{1}{2}]$$

From Feller, (1977, pp. 479–480), these are characteristic functions for discrete distributions. Further, $\phi_{\hat{x}}(t_1) = \phi_{\hat{u}}(t_1)$ on $(-1, 1)$, $(3, 5)$, $(7, 9)$, $\ldots$, and $\phi_{\hat{x}}(t_1) = -\phi_{\hat{u}}(t_1)$ on $(1, 3)$, $(5, 7)$, $(9, 11)$, $\ldots$ Hence the constant changes at every point when the two characteristic functions are zero. However, at these points the derivative of the characteristic function $\phi_{\hat{u}}()$ is not continuous. Thus it is clear that some smoothness restriction is needed.

We next show that with our technical condition the constant $C(r_1)$ is independent of r_1 and equal to 1. Assume not—that is, there are two points $r_1 < s_1$ where $C(r_1) \neq C(s_1)$. Hence, we need only show that the constant cannot change when either (and hence both) characteristic function is (are) zero.

Hence consider a point t_1 such that $\phi_{\hat{u}}(t_1) = 0$. By our technical condition, $\phi_{\hat{u}}(t_1) \neq 0$ and there is a neighborhood $[r_1, s_1]$ where $\phi_{\hat{u}} \neq 0$ (otherwise $\phi_{\hat{u}}(t_1^n) = 0$ for some sequence $t_1^n \to t_1$ with $t_1^n \neq t_1$ and thus $\phi'_{\hat{u}}(t_1) = 0$). Let $C(r_1)$ be the constant for the neighborhood (r_1, t_1) and $C(s_1)$ be the constant for (t_1, s_1). Then we obtain that

$$\phi^N_{N\hat{x}}(t_1) = C(r_1)(N!)\phi'_{\hat{u}}(t_1)$$

by taking the Nth derivative from the left and that

$$\phi^N_{N\hat{x}}(t_1) = C(s_1)(N!)\phi'_{\hat{u}}(t_1)$$

by taking the Nth derivative from the right (notice that we have used the continuity of the derivatives here). Hence the two constants must be equal.

The proof thus works for the region $(-U, U)$. Outside this region the characteristic functions are zero and hence there is nothing to prove.

Hence, we obtain that

$$\phi_{N\hat{x}}(t_1) = \phi_{\hat{u}}(t_1)^N.$$

This condition on the characteristic functions of $N\hat{x}$ and $\hat{u}$ implies that $N\hat{x}$ has the same distribution as the sum of N i.i.d. random variables, each with the same distribution as $\hat{u}$; that is, $\hat{x}$ is an N-fold average of $\hat{u}$. □

Proof of Theorem 3.3. If the distributions have finite second moments, then $\hat{z}\mathrm{E}[\hat{s} \mid \hat{z}] = \lambda_N \hat{z}^2$ because $\mathrm{E}[\hat{s} \mid \hat{z}] = \lambda_N \hat{z}$. Taking expectations with respect to $\hat{z}$ and noting that $\mathrm{Cov}[s, z] = \mathrm{E}[\hat{s}\hat{z}]$, and $\mathrm{Var}[z] = \mathrm{E}[\hat{z}^2]$ yields λ_N. If at least one of the second moments is not finite, then we use the fact that for every characteristic function, ϕ, there exists t_1 and t_2 such that $\phi(t_1) \neq \phi(t_2)$. Let $w_i \equiv \phi_{\hat{s}}(\frac{N}{\lambda_N(N+1)} t_i)$, and define $\psi_{\hat{s}}(w_i) = t_i$. Using (3.3), $\phi_{\hat{u}}(t_i) = w_i$. Define $\psi_{\hat{u}}(w_i) = t_i$ and note that by taking ratios,

$$\frac{N}{\lambda_N(N+1)} = \frac{\psi_{\hat{s}}(w_i)}{\psi_{\hat{u}}(w_i)} \qquad i = 1, 2.$$

Thus, if (3.3) holds, λ_N is uniquely defined. To compute the intercept, note that $\mathrm{E}[\hat{s} \mid \hat{z}] = \lambda_N \hat{z}$ is equivalent to $\mathrm{E}[s - \mathrm{E}[s] \big| z - \mathrm{E}[z]] = \lambda_N(z - \mathrm{E}[z])$ or $\mathrm{E}[s \mid z] = \mathrm{E}[s] - \lambda_N \mathrm{E}[z] + \lambda_N z$. Hence, the intercept term is $\mu_N = \mathrm{E}[s] - \lambda_N \mathrm{E}[z]$. □

Proof of Proposition 3.2. Assuming that $x_j(\hat{s}_j) = a_j + b_j \hat{s}_j$ implies that $\mathrm{E}[X_{\sim i} \mid s_i] = \sum_{j\neq i} a_j + \sum_{j\neq i} b_j \mathrm{E}[\hat{s}_j \mid \hat{s}_i]$. Since $\mathrm{E}[\hat{s}_j \mid \hat{s}_i] = \mathrm{E}[s_j - \mathrm{E}[s_j] \mid \hat{s}_i]$, $s_j = v + \varepsilon_j$, and ε_j is independent of everything,

$$\mathrm{E}[\hat{s}_j \mid \hat{s}_i] = \mathrm{E}[v \mid \hat{s}_i] - \mathrm{E}[v] = \theta + \xi \hat{s}_i - (1 - \xi)\mathrm{E}[v].$$

Substituting into (3.6),

$$x_i(\hat{s}_i) = \left(\frac{1}{2\lambda_N}\right)(\theta + \xi \mathrm{E}[v] - \mu_N - \lambda_N \mathrm{E}[u]) + \frac{\xi}{2\lambda_N}\hat{s}_i - \frac{1}{2}\sum_{j\neq i} a_j - \left(\frac{\xi}{2}\sum_{j\neq i} b_j\right)\hat{s}_i.$$

Thus,

$$b_i = \frac{\xi}{2\lambda_N} - \frac{\xi}{2}\sum_{j\neq i} b_j,$$

$$a_i = \left(\frac{1}{2\lambda_N}\right)(\theta + \xi \mathrm{E}[v] - \mu_N - \lambda_N \mathrm{E}[u]) - \frac{1}{2}\sum_{j\neq i} a_j.$$

Summing over the N strategic traders and solving for a_i and b_i yields

$$x(\hat{s}_i) = \frac{\theta + \xi \mathrm{E}[v] - \mu_N - \lambda_N \mathrm{E}[u]}{\lambda_N(N+1)} + \left(\frac{\xi}{\lambda_N(2 + \xi(N+1))}\right)\hat{s}_i,$$

and so $a_i = a_j \equiv a$ and $b_i = b_j \equiv b$. Since we have assumed that $\mathrm{E}[v \mid z] = \mu_N + \lambda_N z$, $\mathrm{E}[v] = \mu_N + \lambda_N \mathrm{E}[z]$. Using the definition of z, $\mathrm{E}[z] = Na + b\sum_{i=1}^N \mathrm{E}[\hat{s}_i] + \mathrm{E}[u]$. Finally, substitute into the above expression to obtain the version of $x(\hat{s}_i)$ reported in the proposition. □

Proof of Lemma 3.1. By Ferguson's Lemma, $\mathrm{E}[\hat{v} \mid \hat{s}_i] = \xi \hat{s}_i$ if and only if

$$\left.\frac{\partial \phi_{\hat{v},\hat{s}_i}(t_0, t)}{\partial t_0}\right|_{t_0=0} = \xi \frac{\partial \phi_{\hat{s}_i}(t)}{\partial t} \qquad \forall t.$$

Following exactly the same reasoning used in the proof of Theorem 3.1 yields the result. □

Proof of Theorem 3.7. Note that $\hat{v} - \lambda z = \hat{v} - \lambda \sum_{i=1}^N x(s_i) - \lambda\hat{u} = \hat{v} - N\lambda\beta\hat{v} - \lambda\beta\sum_{i=1}^N \epsilon_i - \lambda\hat{u}$ where we are suppressing the N subscript on λ and β. Also, $e^{itz} = e^{it\hat{u}} e^{itN\beta\hat{v}} \Pi_{i=1}^N e^{it\beta\epsilon_i}$. Given this, the derivative condition from Ferguson's Lemma is

$$0 = \mathrm{E}\left[i\big((1 - N\lambda\beta)\hat{v} - \lambda\beta\sum_{i=1}^N \varepsilon_i - \lambda\hat{u}\big)e^{it\hat{u}} e^{itN\beta\hat{v}} \Pi_{i=1}^N e^{it\beta\varepsilon_i}\right].$$

Simplifying,

$$\begin{aligned} 0 = {} & \mathrm{E}\left[i\big(1 - N\lambda\beta\big)\hat{v} e^{itN\beta\hat{v}}\right]\phi_{\hat{u}}(t)\phi_{\epsilon}(\beta t)^N \\ & - \sum_{i=1}^N \mathrm{E}[i\lambda\beta\varepsilon_i e^{it\beta\epsilon_i}]\phi_\epsilon(\beta t)^{N-1}\phi_{\hat{u}}(t)\phi_{\hat{v}}(N\beta t) - \mathrm{E}[i\lambda\hat{u}e^{it\hat{u}}]\phi_{\hat{v}}(N\beta t)\phi_\epsilon(\beta t)^N. \end{aligned}$$

This is equivalent to

$$\frac{d}{dt}\left[\frac{\phi_{\hat{v}}(N\beta t)^{\frac{1-N\beta\lambda}{N\beta}}}{\phi_\epsilon(\beta t)^{N\lambda}\phi_{\hat{u}}(t)^\lambda}\right] = 0 \qquad \forall t.$$

For this derivative to be zero for all t, the numerator must equal the denominator times a constant. Combining this with the fact that $\phi_{\hat{v}}(t) = \phi_\epsilon(t)^{\xi/(1-\xi)}$, yields the version reported in the theorem. □

Proof of Theorem 3.8. An equivalent representation of (3.6) is

$$\phi_{\hat{v}}(t)^{(2-\xi)/\xi} = \phi_{\hat{v}}(t/N)^{(N^2(1-\xi))/\xi}\phi_{\hat{u}}\left(\frac{t}{N\beta_N}\right)^N. \tag{A.1}$$

Our proof relies on showing that $\beta_N\sqrt{N}$ converges to a finite number as N goes to infinity. We begin by supposing that β_N converges to a positive number, possibly infinity. Since $\hat{u}$ has a finite second moment, the Strong Law of Large Numbers implies that

$$\phi_{\hat{u}}\left(\frac{t}{N\beta_N}\right)^N \to 1.$$

Further, since $\hat{v}$ has a finite second moment, by the original Central Limit Theorem,

$$\phi_{\hat{v}}\left(\frac{t}{N}\right)^{N^2} \to e^{-t^2\sigma_v}.$$

Thus the random variable represented by the right-hand side of (A.1) converges to a random variable whose distribution is $\mathcal{N}\big(0, (\sigma_v(1-\xi))/\xi\big)$. But the left-hand side is the characteristic function of a random variable with distribution $\mathcal{N}(0, (\sigma_v(2-\xi))/\xi)$. As these can never be equal, we have that β_N converges to 0 as N goes to infinity.

Given this, we now show that $\beta_N\sqrt{N}$ converges to a finite number. We begin by supposing that it converges to zero. Letting t in (A.1) equal $\beta_N\sqrt{N}\tau$, we note that the last term in (A.1) converges to $e^{-\tau\sigma_u}$ by the original Central Limit Theorem and the first term on the right-hand side converges to 1 by the Strong Law of Large Numbers (and our supposition). Since the left-hand side also converges to 1, we have a contradiction and so $\beta_N\sqrt{N}$ cannot converge to 0. Now, suppose that it diverges. Following the same line of reasoning used in the first paragragh leads to a contradiction.

Finally, suppose that $\beta_N\sqrt{N}$ converges to a finite number, say c. Then, because $t/N\beta_N = (1/(\beta_N\sqrt{N}))(t/(\sqrt{N}))$,

$$\phi_{\hat{u}}\left(\frac{t}{N\beta_N}\right)^N \to e^{-t^2\sigma_u/c^2}$$

and

$$\phi_{\hat{v}}\left(\frac{t}{N}\right)^{N^2} \to e^{-t^2\sigma_v}.$$

Thus, by (A.1),

$$\phi_{\hat{v}}(t) = \exp\left\{\left(\frac{-t^2}{2-\xi}\right)\left(\sigma_v(1-\xi) + \frac{\xi\sigma_u}{\xi}\right)\right\}.$$

Thus v and u must be normally distributed.

Proof of Theorem 4.1. First, since the s_i's have the same distribution, the distribution of s, the distribution of each strategic trader's order is the same and if x is the random variable representing a strategic trader's order, then

$$\mathrm{E}[v \mid z] = \mathrm{P}(z = x \mid z)\mathrm{E}[v \mid x = z] + \mathrm{P}(z = u^* \mid z)\mathrm{E}[v].$$

Second, because $s_i =^d s$, there is a common, induced distribution and density for a strategic trader's order, say H, h and so

$$\mathrm{E}[v \mid z = x] = \mathrm{E}[v \mid \hat{s} = \frac{2\lambda}{\xi}z] = \theta + \xi\mathrm{E}[s] + 2\lambda z$$

and

$$\mathrm{P}(z = x \mid z) = \frac{\rho h(z)}{\rho h(z) + (1-\rho)g(z)},$$

where g is the common density of u_j. Substituting and matching the intercept with μ and the slope with λ implies that

$$\mu = (\theta + \xi\mathrm{E}[s])\mathrm{P}(z = x \mid z) + (1 - \mathrm{P}(z = x \mid z))\mathrm{E}[v] \equiv \mathrm{E}[v]$$
$$\lambda = 2\lambda\mathrm{P}(z = x \mid z).$$

Because $\mathrm{E}[v \mid \hat{s}] = \theta + \xi\mathrm{E}[s] + \xi\hat{s}$ and so $\mathrm{E}[v] = \theta + \xi\mathrm{E}[s] + \xi\mathrm{E}[\hat{s}] = \theta + \xi\mathrm{E}[s]$, $\mu = \mathrm{E}[v]$. The latter implies that $\mathrm{P}(z = x \mid z) = 1/2$ or that $\rho = 1/2$ and $g = h$. Thus, $\hat{x} =^d \hat{u}^*$ or $(\xi/2\lambda)\hat{s} =^d \hat{u}^*$. □

Proof of Theorem 5.2. Since the price function is linear in Z (and z), (3.1) characterizes a strategic trader's best reply and (3.2) the (common) equilibrium strategy. Let Z_j^s be the jth selection of N of the $N+L$ elements of Z and f the conditional probability that Z_j^s is the strategic traders' order vector, $\vec{x}$, given Z. Thus,

$$\mathrm{E}[v \mid Z] = \sum_{j=1}^{K} f(Z_j^s \mid Z)\mathrm{E}[v \mid \vec{x} = Z_j^s],$$

where $K = \binom{N+L}{N}$. Since strategic traders all observe the same signal they place the same order. Thus, if Z_j^s is not a vector of identical orders, say ζ_j, the probability that it is the vector of strategic traders' orders is zero. Hence,

$$\mathrm{E}[v \mid Z] = \mu_N + \left(\frac{(N+1)}{N}\right)\lambda_N \sum_{j=1}^{N+L} f(Z_j^s = \vec{\zeta}_j \mid Z)\zeta_j.$$

By Theorem 5.1, $\mathrm{E}[v \mid Z] = \mathrm{E}[v \mid z]$ and so

$$\left(\frac{(N+1)}{N}\right)\sum_{j=1}^{N+L} f(Z_j^s = \vec{\zeta}_j \mid Z)\zeta_j = \sum_{j=1}^{N+L} z_j \equiv z. \tag{A.2}$$

This condition must hold for all Z so consider an open ball about the set of vectors Z such that $z_i = z_j$ $\forall i, j$. For this measurable set of Z vectors, (A.2) implies that $N+1 = N+L$, which implies that $L = 1$.

Once we know that $L = 1$, all Z vectors that arise with positive probability have N identical elements (the strategic traders' orders) and one additional element, the exogenous trader's order, now labeled z_{N+1} for convenience. If $N \geq 2$, then Z reveals the strategic traders' common order, ζ, with probability one and so $\mathrm{E}[v \mid Z] = \mu_N + \big((N+1)/N\big)\lambda_N\zeta$, which by Theorem 5.1 must equal $\mu_N + \lambda_N(N\zeta + z_{N+1})$. This immediately implies that $N = 1$. Given this, $Z = (z_1, z_2)$ where z_1 is the strategic trader's order with probability $\frac{1}{2}$, which, as in standard Kyle type models, implies that $\mathrm{E}[v \mid Z] = \mu + \lambda z$. □

Proof of Theorem 5.3. If we can show that the splitting rule described ensures that $\mathrm{E}[v \mid Z] = \mathrm{E}[v \mid z]$ then Theorem 5.1 combines with Theorems 3.1 and 3.2 to complete the proof. Let $Z = (Z_j^s, Z_j^u)$ where, for each j, Z_j^s is the jth selection of L elements and Z_j^u is the jth residual L elements of Z. The strategic trade's order and exogenous traders' orders are independent and, given the splitting rule, the probability that Z_j^s is the strategic trader's submitted orders given Z (say $f(Z_j^s \mid Z)$) equals the probability that Z_j^u is the exogenous traders' orders given Z (say $f(Z_j^u \mid Z)$). Thus, using (3.1),

$$\mathrm{E}[v \mid Z] = \sum_{j}^{K} f(Z_j^s \mid Z)\mathrm{E}[v \mid Z_j^s = R] = \mu + \sum_{j}^{K} f(Z_j^s \mid Z)(2\lambda z_j^s),$$

where $K = \binom{2L}{L}$ and z_j^s is the sum of the L elements of Z_j^s. Further, because $f(Z_j^s \mid Z) = f(Z_j^u \mid Z)$,

$$\mathrm{E}[v \mid Z] = \mu + \sum_{j}^{K/2} f(Z_j^s \mid Z)(2\lambda z) = \mu + \lambda z.$$

This last expression is $\mathrm{E}[v \mid z]$ as derived in Section 3. □

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