Detailed Examples of the Modifications to Accommodate any Decimal or Fractional Price Grid

The Holden Model on any Decimal or Fractional Price Grid

This section presents the modifications of the Holden model to accommodate any decimal or fractional price grid. Accounting for decimal price grids adds considerably complexity to the general formulas. I use a standard example of a decimal price grid. Specifically, the possible effective spreads $\left(\text{the } s_j\text{'s}\right)$ are \$0.01, \$0.05, \$0.10, \$0.25, and \$1.00 and $J = 5$.

To build up to the general formulas, I introduce several new variables to keep track of various attributes of a decimal (or fractional) price grid. Let A_j and A_{J+j} be the *total* number of (trade) prices and (no-trade) midpoints, respectively, corresponding to the jth spread $\left(j = 1, 2, \ldots, J\right)$. For prices, there are 100 pennies, 20 nickels, 10 dimes, 4 quarters, and 1 dollar, so $A_1 = 100$, $A_2 = 20$, $A_3 = 10$, $A_4 = 4$, and $A_5 = 1$. For midpoints, there are 100 odd $\frac{1}{2}$ pennies, 20 odd $\frac{1}{2}$ nickels, 10 odd $\frac{1}{2}$ dimes, 4 odd $\frac{1}{2}$ quarters, and 1 odd $\frac{1}{2}$ dollars, so $A_6 = 100$, $A_7 = 20$, $A_8 = 10$, $A_9 = 4$, and $A_{10} = 1$.

To build up this general formula, I need to introduce a new variable.

Define *special* price increments for the jth spread as price increments that can be generated by the jth spread, but not by any larger spreads. Let B_j and B_{J+j} be the number of special prices and special midpoints, respectively, corresponding to the jth spread $\left(j = 1, 2, \ldots, J\right)$. Let D_{jk} be the number of special price increments for the jth spread $\left(j = 1, 2, \ldots, J\right)$ which *overlap* the price increments of the kth spread.

Table 1 summarizes the A_j, B_j, and D_{jk} variables for this standard example.

Table 1 A_j, B_j, and D_{jk} **for a Decimal Price Grid**

j	Corresponding Spread	Prices or midpoints?	A_j	B_j	D_{jk}
1	$0.01	Prices	100	80	$D_{11}=80$
2	$0.05	Prices	20	8	$D_{21}=8, D_{22}=8$
3	$0.10	Prices	10	8	$D_{31}=8, D_{32}=8, D_{33}=8$
4	$0.25	Prices	4	3	$D_{41}=3, D_{42}=3, D_{43}=1, D_{44}=3$
5	$1.00	Prices	1	1	$D_{51}=1, D_{52}=1, D_{53}=1, D_{54}=1, D_{55}=1$
6	$0.01	Midpoints	100	80	$D_{61}=80$
7	$0.05	Midpoints	20	16	$D_{71}=16, D_{72}=16$
8	$0.10	Midpoints	10	10	$D_{81}=0, D_{82}=0, D_{83}=10$
9	$0.25	Midpoints	4	4	$D_{91}=4, D_{92}=4, D_{93}=0, D_{94}=4$
10	$1.00	Midpoints	1	1	$D_{10,1}=0, D_{10,2}=0, D_{10,3}=0, D_{10,4}=0, D_{10,5}=1$

First consider the B_j's for the prices. Out of the 100 penny price increments, 80 are special for the $\$0.01$ spread, because they can be generated by a $\$0.01$ spread, but not by $\$0.05$ spread (or any larger spread). I call these special increments "off pennies." They are all price increments where the last digit is 1, 2, 3, 4, 6, 7, 8, or 9. Out of the 20 nickels, 8 are special for the $\$0.05$ spread, because they can be generated by a $\$0.05$ spread, but not by $\$0.10$ spread (or any larger spread). These "off nickels" are where the last two digits are 05, 15, 35, 45, 55, 65, 85, or 95. Out of the 10 dimes, 8 "off dimes" are special for the $\$0.10$ spread, namely 10, 20, 30, 40, 60, 70, 80, or 90. Out of the 4 quarters, 3 "off quarters" are special for the $\$0.25$ spread, namely 25, 50, or 75. Hence, $B_1=80$, $B_2=8$, $B_3=8$, and $B_4=3$. For the largest $(\text{the } Jth)$ effective spread, all of its price increments are special by definition $B_J=A_J=1$.

Now consider the B_{J+j}'s for the midpoints. Out of 100 odd $\frac{1}{2}$ pennies, 20 are also odd $\frac{1}{2}$ nickels and 80 are special because they are not odd $\frac{1}{2}$ nickels or above. Out of 20 odd $\frac{1}{2}$ nickels, none are odd $\frac{1}{2}$ dimes, but 4 are odd $\frac{1}{2}$ quarters, so 16 odd $\frac{1}{2}$ nickels are special. All 10 of the odd

$\frac{1}{2}$ dimes are special because they are not odd $\frac{1}{2}$ quarters or above. All 4 of the odd $\frac{1}{2}$ quarters are special because they are not odd $\frac{1}{2}$ dollars. The single odd $\frac{1}{2}$ dollar is special, by definition, because it is the highest spread.

First consider the D_{jk}'s for the prices. All 80 special price increments for a $\$0.01$ spread are pennies. All 8 special price increments for a $\$0.05$ spread are both nickels and pennies. All 8 special price increments for a $\$0.10$ spread are dimes, nickels, and pennies. All 3 special price increments for a $\$0.25$ spread are quarters, nickels, and pennies, but importantly one is a dime $(\$0.50)$. The single special price increment for a $\$1.00$ spread is a dollar, quarter, dime, nickel, and penny.

Now consider the D_{jk}'s for the midpoints. All 80 of the special off $\frac{1}{2}$ pennies are off $\frac{1}{2}$ pennies. All 16 of the special off $\frac{1}{2}$ nickels are both off $\frac{1}{2}$ nickels and off $\frac{1}{2}$ pennies. All 10 of the special odd $\frac{1}{2}$ dimes are odd $\frac{1}{2}$ dimes, but none of them are off $\frac{1}{2}$ nickels or off $\frac{1}{2}$ pennies. All 4 of the special odd $\frac{1}{2}$ quarters are odd $\frac{1}{2}$ quarters, off $\frac{1}{2}$ nickels, and off $\frac{1}{2}$ pennies, but none of them are odd $\frac{1}{2}$ dimes. The single special odd $\frac{1}{2}$ dollar is a odd $\frac{1}{2}$ dollar, but it is not an odd $\frac{1}{2}$ quarter, odd $\frac{1}{2}$ dime, off $\frac{1}{2}$ nickel, or off $\frac{1}{2}$ penny.

By creating a tree similar to Figure 1 for a decimal price grid, it is straight-forward to compute the cluster probabilities $\Pr(C_t = j)$ and conditional probability of a half spread given a particular price cluster $\Pr(H_t = h_k \mid C_t = j)$. Using the first part of a tree similar to Figure 1 for a decimal price grid as a template, the probabilities of the trade price clusters are

$$\Pr(C_t = j) = \sum_{k=1}^{j} \gamma_k \mu \frac{D_{jk}}{A_k} \quad j = 1,2,\ldots,J. \tag{A1}$$

Similarly, the probabilities of the no-trade midpoint clusters are

$$\Pr(C_t = J + j) = \sum_{k=1}^{j} \gamma_k (1-\mu) \frac{D_{J+j,k}}{A_{J+k}} \quad \mathrm{j} = 1,2,\ldots,J. \tag{A2}$$

Using the first part of a tree similar to Figure 1 for a decimal price grid as a template, the conditional probability of a half spread given a particular price cluster is

$$\Pr\left(H_t = h_k \mid C_t = j\right) = \frac{\gamma_{|k|}\left(\frac{\mu}{2}\right)^{\frac{D_{j|k|}}{A_{|k|}}}}{\Pr\left(C_t = j\right)} \quad k \neq 0, k \leq j, \text{ and } j = 1, 2, \ldots, J \,, \tag{A3}$$

$$\text{and} \quad \Pr\left(H_t = h_0 \mid C_t = \right) j = 0 \qquad j = 1, 2, \ldots, J \,. \tag{A4}$$

Similarly, the conditional probability of a half spread given a particular midpoint cluster is

$$\Pr\left(H_t = h_0 \mid C_t = J + j\right) = 1 \qquad j = 1, 2, \ldots, J \tag{A5}$$

$$\text{and} \quad \Pr\left(H_t = h_k \mid C_t = J + j\right) = 0 \qquad k \neq 0, k \leq j, \text{ and } j = 1, 2, \ldots, J \,. \tag{A6}$$

To complete the computation, equations (A1) – (A6) are substituted into the likelihood function equation and estimated.

To illustrate how a fractional price grid fits into the general formula of the Holden model, consider a $\$\frac{1}{16}$ price grid where the possible effective spreads are $\$\frac{1}{16}$, $\$\frac{1}{8}$, $\$\frac{1}{4}$, $\$\frac{1}{2}$, and $\$1$. Table 2 shows the A_j, B_j, and D_{jk} variables for this fractional price grid.

Table 2 A_j, B_j, and D_{jk} **for a Fractional Price Grid**

j	Corresponding Spread	Prices or midpoints?	A_j	B_j	O_{jk}
1	$\$\frac{1}{16}$	Prices	16	8	$D_{11} = 8$
2	$\$\frac{1}{8}$	Prices	8	4	$D_{21} = 4, D_{22} = 4$
3	$\$\frac{1}{4}$	Prices	4	2	$D_{31} = 2, D_{32} = 2, D_{33} = 2$
4	$\$\frac{1}{2}$	Prices	2	1	$D_{41} = 1, D_{42} = 1, D_{43} = 1, D_{44} = 1$
5	$1	Prices	1	1	$D_{51} = 1, D_{52} = 1, D_{53} = 1, D_{54} = 1, D_{55} = 1$
6	$\$\frac{1}{16}$	Midpoints	16	16	$D_{61} = 16$
7	$\$\frac{1}{8}$	Midpoints	8	8	$D_{71} = 0, D_{72} = 8$
8	$\$\frac{1}{4}$	Midpoints	4	4	$D_{81} = 0, D_{82} = 0, D_{83} = 4$
9	$\$\frac{1}{2}$	Midpoints	2	2	$D_{91} = 0, D_{92} = 0, D_{93} = 0, D_{94} = 2$
10	$1	Midpoints	1	1	$D_{10,1} = 0, D_{10,2} = 0, D_{10,3} = 0, D_{10,4} = 0, D_{10,5} = 1$

By inspection of the coefficients in Table 2, it is clear that

$$\frac{D_{jk}}{A_k} = \left(\frac{1}{2}\right)^{j-k+1} \quad j = 2,3,\ldots,J-1 \;\text{ and }\; \frac{D_{jk}}{A_k} = \left(\frac{1}{2}\right)^{j-k} \quad j = J\,. \tag{A7}$$

Substituting these coefficients into equations (A1) and (A3) yield the corresponding fractional Holden formulas.

Also, by inspection of the coefficients

$$\frac{D_{J+j,k}}{A_k} = 1 \quad \text{for } k = j \;\text{ and }\; \frac{D_{J+j,k}}{A_k} = 0 \quad \text{for } k < j\,. \tag{A8}$$

Substituting these coefficients into equation (A2) yields the corresponding fractional Holden formula. Equations (A4) – (A6) match the remaining fractional equations, which completes the demonstration.

The Effective Tick Model on any Decimal or Fractional Price Grid

This section presents the general formula for the Effective Tick model, which works on any decimal or fractional price grid.

Many price grids exhibit the property that the price increments overlap 100% between adjacent spread levels. For example in a fractional price grid, all wholes are halves, all halves are quarters, all quarters are eighths, all eighths are sixteenths, etc. However, this property does not hold in general. In the decimal price grid under consideration, all dollars are quarters, all dimes are nickels, and all nickels are pennies, but quarters are different. Two quarters are dimes $(\$0.50,\ \$1.00)$ and two quarters are not dimes $(\$0.25,\ \$0.75)$. The latter two quarters overlap with nickels (two spread layers down), but not dimes (one spread layer down). Hence, there is a need for a relatively elaborate way of tracking these overlaps.

Let O_{jk} be the number of price increments for the jth spread $(j = 1,2,\ldots,J)$ which *overlap* the price increments of the kth spread and do *not overlap* the price increments of any spreads between the jth spread and the kth spread. Similarly, let $O_{J+j,k}$ be the number of overlapping midpoints for the jth spread $(j = 1,2,\ldots,J)$ which *overlap* the midpoints of the kth spread and do *not overlap* the midpoints of any spreads between the jth spread and the kth spread. D_{jk} and O_{jk} are distinct in two ways. First, O_{jk} is the number of *total* price increments which overlap vs. D_{jk} in the number of *special* price increments which overlap. Second, the O_{jk} count *excludes* those increments which overlap the price increments of any spreads between the jth spread and the kth spread vs. the D_{jk} count *includes* those increments. Table 3 summarizes the A_j, B_j, and O_{jk} variables for a standard example of a decimal price grid.

Table 3 A_j, B_j, and O_{jk} **for a Decimal Price Grid**

j	Corresponding Spread	Prices or midpoints?	A_j	B_j	O_{jk}
1	\$0.01	Prices	100	80	
2	\$0.05	Prices	20	8	$O_{21}=20$
3	\$0.10	Prices	10	8	$O_{31}=0, O_{32}=10$
4	\$0.25	Prices	4	3	$O_{41}=0, O_{42}=2, O_{43}=2$
5	\$1.00	Prices	1	1	$O_{51}=0, O_{52}=0, O_{53}=0, O_{54}=1$
6	\$0.01	Midpoints	100	80	
7	\$0.05	Midpoints	20	16	$O_{71}=20$
8	\$0.10	Midpoints	10	10	$O_{81}=0, O_{82}=0$
9	\$0.25	Midpoints	4	4	$O_{91}=0, O_{92}=4, O_{93}=0$
10	\$1.00	Midpoints	1	1	$O_{10,1}=0, O_{10,2}=0, O_{10,3}=0, O_{10,4}=0$

Now consider the $O_{jk}\text{'}s$ for the prices. All 20 nickels are also pennies and all 10 dimes are also nickels. Of the 4 quarters, 2 are dimes and 2 are nickels, but not dimes. The single dollar is also a quarter.

Finally consider $O_{J+j,k}\text{'}s$ for the midpoints. All 20 of the odd $\frac{1}{2}$ nickels overlap with odd $\frac{1}{2}$ pennies. None of the odd $\frac{1}{2}$ dimes overlap with odd $\frac{1}{2}$ nickels or odd $\frac{1}{2}$ pennies. None of the odd $\frac{1}{2}$ quarters overlap with odd $\frac{1}{2}$ dimes, but all 4 of them overlap with odd $\frac{1}{2}$ nickels. The odd $\frac{1}{2}$ dollar doesn't overlap with anything below.

Given all of the infrastructure variables in Table 3, the general formula of the Effective Tick model can be stated. The general formula for the unconstrained probability of the jth spread is

$$U_j = \begin{cases} \left(\dfrac{A_1}{B_1}\right)F_1 + \left(\dfrac{A_{J+1}}{B_{J+1}}\right)F_{J+1} & j=1 \\ \left(\dfrac{A_j}{B_j}\right)F_j - \displaystyle\sum_{k=1}^{j-1}\left(\dfrac{O_{jk}}{B_k}\right)F_k + \left(\dfrac{A_{J+j}}{B_{J+j}}\right)F_{J+j} - \displaystyle\sum_{k=1}^{j-1}\left(\dfrac{O_{J+j,k}}{B_{J+k}}\right)F_{J+k} & j=2,3,\ldots,J \end{cases} . \quad \text{(A9)}$$

The rest of the effective tick computation is the same as the fraction grid case.

To illustrate how a fractional price grid fits into the general formula of the Effective Tick model, consider a $\$\frac{1}{16}$ price grid where the possible effective spreads are $\$\frac{1}{16}$, $\$\frac{1}{8}$, $\$\frac{1}{4}$, $\$\frac{1}{2}$, and $\$1$. Table 4 shows the A_j, B_j, and O_{jk} variables for this fractional price grid.

Table 4 A_j, B_j, and O_{jk} **for a Fractional Price Grid**

j	Corresponding Spread	Prices or midpoints?	A_j	B_j	O_{jk}
1	$\$\frac{1}{16}$	Prices	16	8	
2	$\$\frac{1}{8}$	Prices	8	4	$O_{21}=8$
3	$\$\frac{1}{4}$	Prices	4	2	$O_{31}=0, O_{32}=4$
4	$\$\frac{1}{2}$	Prices	2	1	$O_{41}=0, O_{42}=0, O_{43}=2$
5	$1	Prices	1	1	$O_{51}=0, O_{52}=0, O_{53}=0, O_{54}=1$
6	$\$\frac{1}{16}$	Midpoints	16	16	
7	$\$\frac{1}{8}$	Midpoints	8	8	$O_{71}=0$
8	$\$\frac{1}{4}$	Midpoints	4	4	$O_{81}=0, O_{82}=0$
9	$\$\frac{1}{2}$	Midpoints	2	2	$O_{91}=0, O_{92}=0, O_{93}=0$
10	$1	Midpoints	1	1	$O_{10,1}=0, O_{10,2}=0, O_{10,3}=0, O_{10,4}=0$

Consider the coefficients of equation (A9). From Table 4, it is clear that

$$\frac{A_j}{B_j}=2 \ \text{ for } j=1,2,\ldots,J-1, \quad \frac{A_J}{B_J}=1, \text{ and } \frac{A_{J+j}}{B_{J+j}}=1 \ \text{ for } j=1,2,\ldots,J . \quad \text{(A10)}$$

In fractional price grids, the price increments exhibit 100% overlap between adjacent spread levels (i.e., all wholes are halves, all halves are quarters, all quarters are eighths, etc.). This implies that

$$\frac{O_{jk}}{B_k} = 1 \text{ for k=j-1 and } \frac{O_{jk}}{B_k} = 0 \text{ for all } k < j-1 \quad \text{and for } j = 2,3,\ldots,J \,. \qquad \text{(A11)}$$

In fractional price grids, the midpoint clusters exhibit 0% overlap between adjacent spread levels (i.e., no odd $\frac{1}{2}$ midpoints are odd $\frac{1}{4}$ midpoints, no odd $\frac{1}{4}$ midpoints are odd $\frac{1}{8}$ midpoints, no odd $\frac{1}{8}$ midpoints are odd $\frac{1}{16}$ midpoints, etc.). This implies that

$$\frac{O_{J+j,k}}{B_{J+k}} = 0 \qquad \text{for all } k \text{ and for } j = 2,3,\ldots,J \,. \qquad \text{(A12)}$$

Substituting the fractional price grid coefficients in equations (A10), (A11), and (A12) into the general Effective Tick formula (A9) yields the fractional Effective Tick formula.