Nonsequential Search Equilibrium with Search Cost Heterogeneity*

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Abstract

We generalize the model of Burdett and Judd (1983) to the case where an arbitrary finite number of firms sells a homogeneous good to buyers who have heterogeneous search costs. We show that a price dispersed symmetric Nash equilibrium always exists. Numerical results show that the behavior of prices and consumer surplus with respect to the number of firms hinges upon the nature of search cost dispersion: when search costs are relatively concentrated, entry of firms leads to lower average prices and greater consumer surplus; however, for relatively dispersed search costs, the mean price goes up and consumer surplus may decrease with the number of firms.

Keywords: nonsequential search, entry, oligopoly, arbitrary search cost distributions
JEL Classification: D43, D83, C72

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1 Introduction

The theory of search has become a toolkit for the understanding of the role of informational imperfections in generating observed market inefficiencies. Burdett and Judd's (1983) model of nonsequential search is one of the seminal contributions. Burdett and Judd show that price dispersion can arise as an equilibrium phenomenon in environments where firms and consumers are rational and identical. Burdett and Judd's model of nonsequential search has seen a number of important extensions, including McAfee's (1995) study of multiproduct firms, Fershtman and Fishman's (1992) study of price dynamics, Acemoglu and Shimer's (2000) study of a general equilibrium labour market, and Janssen and Moraga-González's (2004) study of oligopolistic pricing. In labor economics, Burdett and Mortensen's (1998) model has become a canonical framework for explaining wage dispersion and turnover.

This paper generalizes the nonsequential search model studied in Burdett and Judd (1983) to the case in which consumers have heterogeneous search costs. Even though the existence and characterization of price dispersed equilibria has not yet been shown, this extension has been used in the empirical literature as the workhorse model for structurally estimating search costs in homogenous product markets. By exploiting the equilibrium conditions of the search model, Hong and Shum (2006) were the first to show that search costs can be recovered from price data only. Subsequent empirical work extended the methodology by improving the estimation method (Moraga-González and Wildenbeest, 2008; Miessi Sanches, Silva Junior, and Srisuma, 2015), and showed how vertical product differentiation (Wildenbeest, 2011), data from different markets (Moraga-González, Sándor, and Wildenbeest, 2013), forward looking consumers (Blevins and Senney, 2016), search data (De los Santos, 2012), and quantity data (Zhang, Chan, and Xie, 2015) can be incorporated into the framework. Moreover, the methodology has been applied in many different settings, including grocery stores (Richards, Hamilton, and Allender, 2016; González and Miles, 2015) and gasoline markets (Nishida and Remer, 2015).

This paper contributes to this literature by demonstrating that optimal firm and consumer behavior can be integrated in such a way that the market equilibrium can be described by an N-dimensional nonlinear system of equations. This is useful for two reasons. First, it provides us with a simple way to simulate the market equilibrium and, second, it enables us to address the existence of equilibrium issue using a fixed point argument. Our main theorem shows that an equilibrium always exists for arbitrary search cost distributions with strictly increasing cumulative distribution function (CDF). Regarding the uniqueness of a symmetric Nash equilibrium in mixed strategies, in contrast to the models of Burdett and Judd (1983) and Janssen and Moraga-González (2004), we believe that consumer search cost heterogeneity results in a unique equilibrium when the search cost density is not too decreasing. In fact, we are able to show that there exists only one symmetric equilibrium when $N = 2$ and search costs follow a power distribution, no matter whether the density is increasing or decreasing. Though this result proves very difficult to extend to markets with an arbitrary number of firms, or with different search costs distributions, our proof suggests that uniqueness should also obtain with densities that are not too decreasing.

The paper also studies how the number of firms affects equilibrium pricing as well as consumer and total surplus. Though exceptions exist, the standard view in Industrial Organization is that more firms will result in an increase in consumer surplus. For example, in Cournot models with homogeneous products, Seade (1980) shows that output, and hence consumer surplus, typically increases with entry. With price competition, Salop's (1979) well-known model of localized competition with differentiated products shows that price and consumer surplus increases with the number of competitors. More general models of price competition such as Perloff and Salop (1985) and Chen and Riordan (2008) show that even if price increases after entry, consumer gains from greater product variety will typically dominate so that consumers surplus will increase.

What is common to these models is that all the products and the prices of the firms are perfectly visible for consumers. In more realistic settings, consumers have to search for prices.

Stahl (1989) demonstrates that entry may weaken competition in sequential search models for homogeneous products. A similar result obtains in the nonsequential search model of Janssen and Moraga-González (2004). With differentiated products, Anderson and Renault (1999) show that an increase in the number of competitors lowers price and increases consumer surplus. More recently, Chen and Zhang (2016) demonstrate that entry may result in lower consumer surplus once we allow for firm asymmetries.

Our main addition to this literature is to show that whether entry strengthens or weakens competition depends on how dispersed search costs are. As far as we know, this observation has not been made before. In fact, when consumers have similar search costs, in our model with search cost heterogeneity mean prices fall and consumer surplus increases in the number of firms. By contrast, if search costs are relatively dispersed across the consumer population, mean prices increase in N and consumer surplus decreases in N provided that N is initially sufficiently high. The crucial distinction between the two cases is that with very little heterogeneity, firms have to keep their prices appealing for all consumers as the number of firms goes up. In contrast, with a lot of heterogeneity consumers differ substantially in their search intensities and those who conduct an exhaustive search in the market become disproportionately less attractive for a firm as more competitors are around. As a result, firms focus more on the higher search cost consumers and raise prices on average.[1]

Our finding that prices may increase or decrease depending on how dispersed search costs are, reveals that allowing for search cost heterogeneity is important when trying to explain empirical results from markets in which search frictions are deemed significant and many sellers are active. For instance, Baye, Morgan, and Scholten (2004) find a negative relationship between average price and the number of competing firms for consumer electronics products on a price comparison site, and Barron, Taylor, and Umbeck (2004) find a modest but significant negative effect of the

[1] The latter result is in line with Chen and Zhang (2016) and suggests a similarity between their model with a single price and product differentiation and our model with a single product and price differentiation.

number of sellers on average prices in the gasoline market.[2] While these empirical findings are inconsistent with simpler theoretical search models such as Varian (1980), Janssen and Moraga-González (2004), and Stahl (1989), they can be rationalized by our model as long as search costs are not very dispersed.

The structure of this paper is as follows. In the next section, we present the nonsequential consumer search model studied here and discuss existence and uniqueness of a price dispersed symmetric equilibrium. In Section 3 we present simulation results illustrating the effects of an increase in the number of firms. We conclude in Section 4. All the proofs are placed in the Appendix to ease the reading.

2 The model

We examine an *oligopolistic* version of Burdett and Judd (1983) with consumer *search cost heterogeneity*. A total of N firms produce a good at unit cost r. There is a unit mass of buyers. Each consumer inelastically demands one unit of the good and is willing to pay for the good a maximum of v. Consumers search for prices nonsequentially and buy from the cheapest store in their sample. Obtaining price quotations, including the first, is costly. Search costs differ across consumers. A buyer's search cost is drawn independently from a common atomless distribution $G(c)$ with support $(0, \infty)$ and positive density $g(c)$ everywhere. A consumer with search cost c sampling k firms incurs a total search cost kc.

Firms and buyers play a simultaneous moves game. An individual firm chooses its price taking rivals' prices as well as consumers' search behavior as given. A firm i's strategy is denoted by a distribution of prices $F_i(p)$. Let $F_{-i}(p)$ denote the vector of prices charged by firms other than i. The (expected) profit to firm i from charging price p_i given rivals' strategies is denoted $\Pi(p_i, F_{-i}(p))$. Likewise, an individual buyer takes as given firm pricing and decides on his/her optimal search

[2] Other studies that find a negative relation between average price and the number of sellers are Haynes and Thompson (2008) for digital cameras that appear on a price comparison site and Lewis (2008) and Lach and Moraga-González (2009) for gasoline markets.

strategy to maximize his/her expected utility. The strategy of a consumer with search cost c is then a number k of prices to sample. Let the fraction of consumers sampling k firms be denoted by μ_k. We shall concentrate on symmetric Nash equilibria. A symmetric equilibrium is a distribution of prices $F(p)$ and a collection $\{\mu_0, \mu_1, \ldots, \mu_N\}$ such that (a) $\Pi_i(p, F_{-i}(p))$ is equal to a constant $\overline{\Pi}$ for all p in the support of $F(p)$, $\forall i$; (b) $\Pi_i(p, F_{-i}(p)) \leq \overline{\Pi}$ for all p, $\forall i$; (c) a consumer sampling k firms obtains no lower utility than by sampling any other number of firms; and (d) $\sum_{k=0}^{N} \mu_k = 1$. Let us denote the equilibrium density of prices by $f(p)$, with maximum price $\overline{p}$ and minimum price $\underline{p}$.

Next we present our results on existence and uniqueness of symmetric equilibrium in mixed strategies. Before stating our main theorem, we re-state some of the results in Burdett and Judd (1983) in Propositions 1-4. Since these propositions are minor modifications of results in Burdett and Judd we do not present formal proofs. We only claim novelty of Propositions 5 and 6 and Theorem 1.

We start by indicating that, for an equilibrium to exist, there must be some consumers who search just once and others who search more than once.

Proposition 1 *If a symmetric equilibrium exists, then* $1 > \mu_1 > 0$ *and* $\mu_k > 0$ *for some* $k = 2, 3, \ldots, N$.

The intuition behind this result is simple. Suppose all consumers did search at least twice; then all firms would be subject to price comparisons with rival firms so firm pricing would be competitive. This however is contradictory because then consumers would not be willing to search that much in the first place. Suppose now that no consumer did compare prices; then firms would charge the monopoly price. This is also contradictory because in that case consumers would not be willing to search at all.[3]

[3] In the original model of Burdett and Judd (1983) there always exists an equilibrium where all firms charge the monopoly price. This is because it is assumed that consumers have a reservation price p^* at which consumer surplus is big enough to cover the search cost.

We next observe that, given consumer behavior, for an equilibrium to exist it must be the case that firm pricing is characterized by mixed strategies.

Proposition 2 *If a symmetric equilibrium exists, $F(p)$ must be atomless with upper bound equal to v.*

That dispersion must arise is easily understood. If a particular price is chosen with strictly positive probability then a deviant can gain by undercutting such a price. This competition for the price-comparing consumers cannot drive the price down zero since then a deviant would prefer to raise its price and sell to the consumers who do not compare prices.

We now turn to consumers' search behavior. Expenditure minimization requires a consumer with cost c to continue to draw prices from the price distribution $F(p)$ till the expected gains of searching one more time fall below her search cost. The expected gains from searching $k+1$ prices rather than k prices are given by $E[\min\{p_1, p_2, \ldots, p_{k+1}\}] - E[\min\{p_1, p_2, \ldots, p_k\}]$, where E denotes the expectation operator. These gains are strictly positive, decreasing and convergent to zero (see MacMinn, 1980). As a result, a consumer with search cost c will choose to sample k firms provided that the following three inequalities hold:

$$\begin{aligned} v - E[\min\{p_1, p_2, \ldots, p_k\}] - kc > 0; \\ E[\min\{p_1, p_2, \ldots, p_{k-1}\}] - E[\min\{p_1, p_2, \ldots, p_k\}] > c; \\ E[\min\{p_1, p_2, \ldots, p_{k+1}\}] - E[\min\{p_1, p_2, \ldots, p_k\}] < c. \end{aligned}$$

Since the search cost distribution $G(c)$ has support $(0, \infty)$ and positive density everywhere, there exists a consumer indifferent between not searching at all and searching once. Let the search cost of this consumer be denoted c_0. Then

$$c_0 = v - E[p], \tag{1}$$

since the expected surplus for a consumer who searches one time is $v - E[p]$. Consumers for whom $c \geq c_0$ obtain negative surplus if they search. As a result, the share of consumers who do not participate in the market altogether is $\mu_0 = \int_{c_0}^{\infty} dG(c) > 0$. Likewise, let c_k be the search cost of the consumer indifferent between searching k times and searching $k+1$ times:

$$c_k = E[\min\{p_1, p_2, \ldots, p_k\}] - E[\min\{p_1, p_2, \ldots, p_{k+1}\}], \; k = 1, 2, \ldots, N-1. \tag{2}$$

Consumers for whom $c_k \leq c \leq c_{k-1}$ search k times. As a result $\mu_k = \int_{c_k}^{c_{k-1}} dG(c) > 0$, $k = 2, 3, \ldots, N$ with $c_N = 0$. The following result summarizes:

Proposition 3 *Given any atomless price distribution $F(p)$, optimal consumer search behavior is characterized as follows: consumers whose search cost $c \leq c_{N-1}$ search for N prices, consumers whose search cost $c \in [c_k, c_{k-1}]$ search for k prices, $k = 1, 2, \ldots, N-1$, and consumers whose search cost $c \geq c_0$ stay out of the market, where c_k, $k = 0, 1, 2, \ldots, N-1$, is given by equations (1) and (2).*

Proposition 3 shows that for any given atomless price distribution optimal consumer search leads to a unique grouping of consumers.

We now examine the pricing behavior of the firms. Following Burdett and Judd (1983), the expected profit to firm i from charging price p_i when its rivals draw a price from the CDF $F(p)$ is

$$\Pi_i(p_i; F(p)) = (p_i - r) \left(\sum_{k=1}^{N} \frac{k}{N} \mu_k (1 - F(p_i))^{k-1} \right).$$

This expression arises because, given consumer search strategies, a firm i charging p_i sells to a consumer who compares k prices whenever the price of the other $k-1$ firms is higher than p_i, which happens with probability $(1 - F(p_i))^{k-1}$.

In equilibrium, a firm must be indifferent between charging any price in the support of $F(p)$ and charging the upper bound $\overline{p}$. Thus, any price in the support of $F(p)$ must satisfy $\Pi_i(p_i; F(p)) =$

$\Pi_i(\overline{p}; F(p))$. Since $\Pi_i(\overline{p}; F(p))$ is monotonically increasing in $\overline{p}$, it must be the case that $\overline{p} = v$. As a result, equilibrium requires

$$(p_i - r)\left[\sum_{k=1}^{N} k\mu_k(1 - F(p_i))^{k-1}\right] = \mu_1(v - r). \tag{3}$$

Unfortunately, this equation cannot be solved for $F(p_i)$ analytically (except in special cases). However, one can prove existence of an equilibrium price distribution $F(p_i)$. Let us rewrite equation (3) as follows:

$$\sum_{k=1}^{N} k\mu_k(1 - F(p_i))^{k-1} = \frac{\mu_1(v - r)}{(p_i - r)}. \tag{4}$$

Note that the RHS of equation (4) is positive and does not depend on $F(p_i)$. By contrast, since $F(p_i)$ must take values on $[0, 1]$, the LHS of equation (4) is a positive-valued function that decreases in $F(p_i)$ monotonically. At $F(p_i) = 0$, the LHS takes on value $\sum_{k=1}^{N} k\mu_k$, while at $p_i = v$ it takes on value μ_1. As a result, for every price $p_i \in (\underline{p}, v)$, there is a unique solution to equation (4) satisfying $F(p_i) \in [0, 1]$; moreover, the solution $F(p_i)$ is monotonically increasing in p_i. The following result summarizes these findings.

Proposition 4 *Given consumer search behavior $\{\mu_k\}_{k=0}^{N}$, there exists a unique symmetric equilibrium price distribution $F(p)$. In equilibrium firms charge prices randomly chosen from the set $\left[\frac{\mu_1(v-r)}{\sum_{k=1}^{N} k\mu_k} + r, v\right]$ according to the price distribution defined implicitly by equation (3).*

Proposition 4 shows that the equilibrium price distribution is unique for any given grouping of consumers. For the price distribution in Proposition 4 to be an equilibrium of the game, the conjectured grouping of consumers has to be the outcome of optimal consumer search. This requires that the following system of equations holds:

$$\mu_k = \int_{c_k}^{c_{k-1}} dG(c), \text{ for all } k = 1, 2, \ldots, N - 1; \tag{5}$$

$$\mu_N = \int_{0}^{c_{N-1}} dG(c), \tag{6}$$

with $\mu_0 = 1 - \sum_{k=1}^{N} \mu_k$ and where c_0 and $c_k, k = 1, 2, \ldots, N-1$ are the solutions to

$$c_0 = v - E[p]; \tag{7}$$

$$c_k = E[\min\{p_1, p_2, \ldots, p_k\}] - E[\min\{p_1, p_2, \ldots, p_{k+1}\}], \; k = 1, 2, \ldots, N-1, \tag{8}$$

where the expectation operator is taken over the distribution of prices which solves equation (3).

Using the distributions of the order statistics, and after successively integrating by parts, we can rewrite equations (7) and (8) as follows:

$$c_0 = \int_{\underline{p}}^{v} F(p)dp; \tag{9}$$

$$c_k = \int_{\underline{p}}^{v} F(p)(1 - F(p))^k dp, \; k = 1, 2, \ldots, N-1. \tag{10}$$

$F(p)$ is monotonically increasing in p so we can use equation (3) to find its inverse:

$$p(z) = \frac{\mu_1(v - r)}{\sum_{k=1}^{N} k\mu_k(1 - z)^{k-1}} + r. \tag{11}$$

Using this inverse function, integration by parts and the change of variables $z = F(p)$ in equations (9) and (10) yields:

$$c_0 = v - \int_0^1 p(z)dz; \tag{12}$$

$$c_k = \int_0^1 p(z)[(k+1)z - 1](1 - z)^{k-1}dz, \; k = 1, 2, \ldots, N-1. \tag{13}$$

Therefore we can state that:

Proposition 5 *If a symmetric equilibrium of the game exists then consumers search according to Proposition 3, firms set prices according to Proposition 4, and the series of critical cutoff points*

$\{c_k\}_{k=0}^{N-1}$ is given by the solution to the system of equations:

$$c_0 = (v-r)\left(1-\int_0^1 \frac{G(c_0)-G(c_1)}{\sum_{k=1}^{N} k[G(c_{k-1})-G(c_k)]u^{k-1}}du\right); \tag{14}$$

$$c_k = (v-r)\int_0^1 \frac{[G(c_0)-G(c_1)]\left[ku^{k-1}-(k+1)\,u^k\right]}{\sum_{k=1}^{N} k[G(c_{k-1})-G(c_k)]u^{k-1}}du,\ \ k=1,2,...,N-1. \tag{15}$$

This result is useful for two reasons. First, it provides a straightforward way to compute and simulate the market equilibrium. For fixed v, r, and $G(c)$, the system of equations (14)–(15) can be solved numerically. If a solution exists, then the consumer equilibrium is given by equations (5)–(6) and the price distribution follows readily from equation (11). Secondly, this result enables us to address the existence and uniqueness of equilibrium issues, which are the subject of our next statement.

Theorem 1 *For any consumer valuation v and firm marginal cost r such that $v > r \geq 0$ and for any search cost distribution function $G(c)$ with support $(0,\infty)$ such that either $g(0) > 0$ or $g(0) = 0$ and $g'(0) > 0$, a symmetric equilibrium exists in a market with an arbitrary number of firms N.*

The proof of this result, which is in the Appendix, builds on Brouwer's fixed point theorem. To apply the theorem, we first construct an auxiliary mapping and show that a market equilibrium is given by a fixed point of such a mapping. A difficulty we encounter in applying Brouwer's fixed point theorem directly is that the auxiliary mapping happens to be discontinuous at zero. This would not be a problem if we could bound the domain of definition of the auxiliary mapping. However, it is not possible to find a bound of the domain of definition that is appropriate for arbitrary search cost distributions. Because of this, we modify the auxiliary mapping in the neighbourhood of 0 and apply Brouwer's fixed point theorem to the modified auxiliary mapping.

Proposition 6 *When $N = 2$ and search costs follow a power distribution $G(c) = (c/\overline{c})^a$, $a > 0$, the symmetric equilibrium is unique.*

This result establishes uniqueness of equilibrium when the market is operated by two firms and search costs follow a power distribution.[4] General results on uniqueness prove to be very difficult to obtain because we cannot compute the equilibrium explicitly. However, the proof of this Proposition clearly suggests that uniqueness should hold when the density of search cost is not too decreasing, but of course this is not necessary as with power functions with parameter $a < 1$. Moreover, simulations of the model for different parameters and search cost distributions, some of which we present below in Section 3, suggest the uniqueness result is more general. This points towards lack of consumer search cost heterogeneity being the driver of multiplicity in the simpler models of Burdett and Judd (1983) and Janssen and Moraga-González (2004).[5]

3 Price equilibrium and the number of firms

In this section we study how an increase in the number of competitors affects pricing and consumer surplus for different search cost distributions.[6] As mentioned in Section 1, though exceptions exist, the standard view in Industrial Organization is that more firms will result in an increase in consumer surplus. The price and welfare effects of entry in our model are difficult to derive analytically since the equilibrium price distribution cannot be obtained in closed form. We therefore proceed by solving the model numerically.

Consider a market in which consumers' search costs follow a beta distribution, with parameters (α, β). Consumers valuations v are identical; we set $v = 1.25$. The firms' marginal cost r is set equal to 0. In what follows, we fix the mean search cost to 0.5 and compare how the market works for two different levels of search cost dispersion, using the two search cost densities plotted in Figure 1. In particular, we focus on the effects of entry on prices and surplus and study how these effects depend on the amount of search cost dispersion. Our objective is to study how changes in the

[4] We note that the existence result of Theorem 1 also holds for bounded support provided that the upper bound is sufficiently large so that in equilibrium $\mu_0 > 0$.

[5] Notice that these papers have discrete distributions of search costs, which can be interpreted as densities that increase and decrease extremely rapidly.

[6] Janssen and Moraga-González (2004) study the effects of entry in a model with a two-point search cost distribution that includes an atom of shoppers.

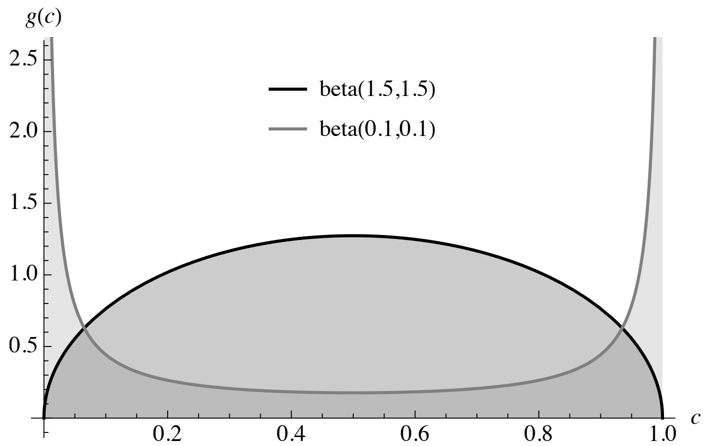


Figure 1: Beta distribution for different parameter values

number of firms lead to changes in prices and welfare *taking as given the search cost distribution*, and not to compare price and welfare levels *across* different search cost distributions.[7]

Table 1: Equilibrium search intensities for $(\alpha, \beta) = (1.5, 1.5)$ (mean is 0.5; variance is 0.06)

Search	Number of firms N													
intensity	2	3	4	5	6	7	8	9	10	11	12	13	14	15
μ_0	0.86	0.82	0.80	0.80	0.80	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79
μ_1	0.12	0.15	0.15	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16
μ_2	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
μ_3	-	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_4	-	-	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
μ_5	-	-	-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
μ_6	-	-	-	-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
μ_7	-	-	-	-	-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
μ_8	-	-	-	-	-	-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
μ_9	-	-	-	-	-	-	-	0.00	0.00	0.00	0.00	0.00	0.00	0.00
μ_{10}	-	-	-	-	-	-	-	-	0.00	0.00	0.00	0.00	0.00	0.00
μ_{11}	-	-	-	-	-	-	-	-	-	0.00	0.00	0.00	0.00	0.00
μ_{12}	-	-	-	-	-	-	-	-	-	-	0.00	0.00	0.00	0.00
μ_{13}	-	-	-	-	-	-	-	-	-	-	-	0.00	0.00	0.00
μ_{14}	-	-	-	-	-	-	-	-	-	-	-	-	0.00	0.00
μ_{15}	-	-	-	-	-	-	-	-	-	-	-	-	-	0.00

Notes: The entries with zeros are not exactly zeros but very small strictly positive numbers.

We start with a market where search cost dispersion is relatively low. For this we set $(\alpha, \beta) = (1.5, 1.5)$.[8] Given the parameters of the model, we solve for the equilibrium of the model for

[7]Moraga-González, Sándor, and Wildenbeest (2016) study how the shape of the search cost distribution affects pricing in a model of consumer search for differentiated products and find that as search costs increase, prices may go up or down depending on the exact shape of the search cost density.

[8]Mean search cost is equal to $\frac{\alpha}{\alpha+\beta} = 0.5$ and variance is equal to $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \simeq 0.06$.

different numbers of firms. The results are reported in Table 1. Table 1 shows how consumer search intensities change as we increase the number of firms. As shown in the table, an important feature is that very few consumers make an exhaustive search in the market. For instance, if there are 10 firms in the industry about 99% of the consumers searches for a maximum of 3 firms. Moreover, although all search intensities are strictly positive, hardly any consumer searches for more than 4 firms, even in markets where there are more than 10 firms that can be visited.

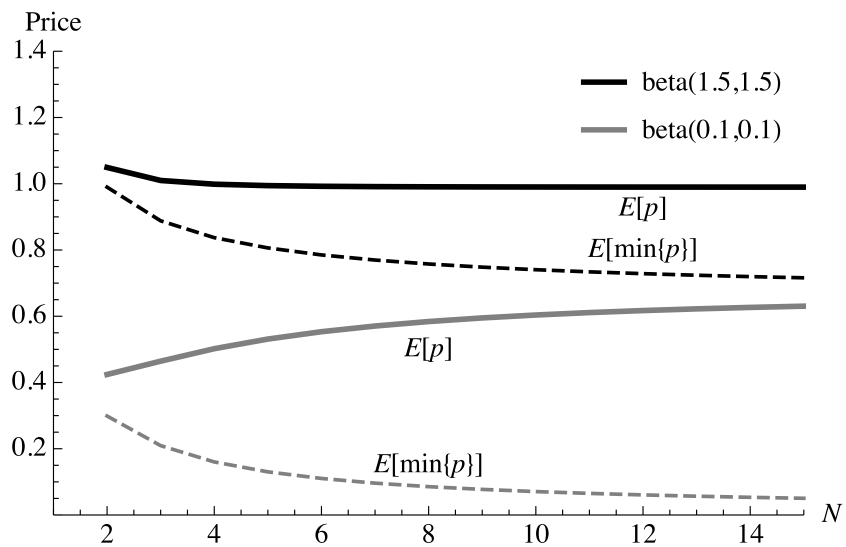
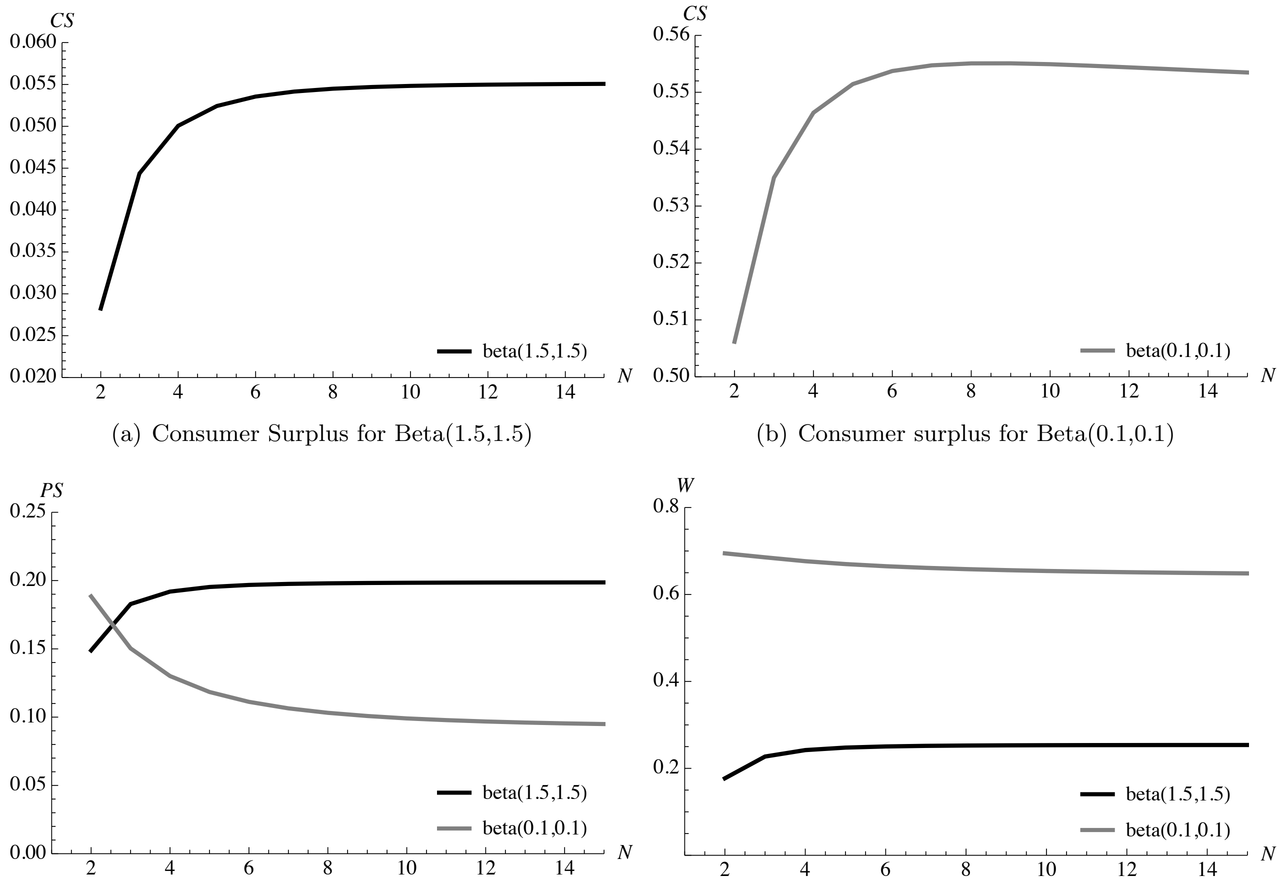


Figure 2: Prices and the number of firms

The fact that most consumers do not compare many prices is reflected in equilibrium prices. Figure 2 shows how prices change with the number of firms. As indicated by the black solid line, the average price is relatively high under duopoly but decreases as the number of firms rises. The decrease of the mean price is due to the fact that the share of consumers comparing two or three prices increases in the number of firms. The average price is what is important for consumers who do not exercise price comparisons so consumers benefit from the resulting average price decreases. The dashed black line in Figure 2 shows that the expected minimum price also decreases as N increases, which is relevant for consumers who compare prices of all firms in the market.[9] These gains are

[9] The expected minimum price out of s searches is

$$E[\min\{p_1, p_2, \ldots, p_s\}] = \int_0^1 \frac{vs\mu_1}{\sum_{k=1}^{N} k\mu_k(1-z)^{k-s}} dz.$$



(a) Consumer Surplus for Beta(1.5,1.5)

(b) Consumer surplus for Beta(0.1,0.1)

(c) Producer surplus

(d) Welfare

Figure 3: Surplus and the number of firms

also reflected in the consumer surplus, indicated in Figure 3(a), as it increases in N.[10] The black curves in Figures 3(c) and 3(d) show that aggregate profits, which are given by $PS = (v - r)\mu_1$, and social welfare are also increasing in N. Social welfare is just the difference between consumers' valuation and the unit cost minus total search cost—even though this shows that search costs are wasteful, the consumers who search more when N goes up are exactly those with low search costs, and as a result the total search cost in the market goes down and welfare goes up in N. In sum, entry in this case of low search cost dispersion would lead to lower average prices, higher consumer surplus, lower industry profits, and higher welfare.

[10] Aggregate consumer surplus is given by $CS = \sum_{s=1}^{N} \mu_s \left[v - E[\min\{p_1, p_2, \ldots, p_s\} - s \cdot \overline{c}_s\right]$, where the total average cost of searching s times is $s \cdot \overline{c}_s = s \int_{c_s}^{c_{s-1}} cg(c)dc$.

The situation is quite different when search costs are much more dispersed, holding everything else equal. Consider $(\alpha, \beta) = (0.1, 0.1)$, which implies the new search cost distribution is a mean-preserving spread of the previous one. The new equilibrium search intensities are reported in Table 2. What is different in this case of high search cost dispersion is that a great deal of consumers conduct an exhaustive search; as before, the extent of price comparison in the market increases as the number of firms rises.

Table 2: Equilibrium search intensities for $(\alpha, \beta) = (0.1, 0.1)$ (mean is 0.5; variance is 0.21)

	Number of firms N													
	2	3	4	5	6	7	8	9	10	11	12	13	14	15
μ_0	0.43	0.44	0.45	0.46	0.46	0.47	0.47	0.47	0.47	0.47	0.48	0.48	0.48	0.48
μ_1	0.15	0.12	0.10	0.09	0.09	0.09	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
μ_2	0.42	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.03	0.03
μ_3	-	0.39	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.02
μ_4	-	-	0.37	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
μ_5	-	-	-	0.36	0.02	0.02	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.01
μ_6	-	-	-	-	0.34	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_7	-	-	-	-	-	0.33	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_8	-	-	-	-	-	-	0.33	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_9	-	-	-	-	-	-	-	0.32	0.01	0.01	0.01	0.01	0.01	0.01
μ_{10}	-	-	-	-	-	-	-	-	0.31	0.01	0.01	0.01	0.01	0.01
μ_{11}	-	-	-	-	-	-	-	-	-	0.31	0.01	0.01	0.01	0.01
μ_{12}	-	-	-	-	-	-	-	-	-	-	0.30	0.01	0.01	0.01
μ_{13}	-	-	-	-	-	-	-	-	-	-	-	0.30	0.01	0.01
μ_{14}	-	-	-	-	-	-	-	-	-	-	-	-	0.29	0.01
μ_{15}	-	-	-	-	-	-	-	-	-	-	-	-	-	0.29

Notes: The entries with zeros are not exactly zeros but very small strictly positive numbers.

Figure 2 also plots the equilibrium mean price as well as the expected minimum price against the number of competitors in the industry for the case of relatively dispersed search costs (gray curves). Under duopoly, the average price is lower in comparison to the case in which search costs are not very dispersed. What is remarkably different is that the mean price increases as more firms enter the industry. Moreover, Figure 3(b) shows that consumer surplus first increases and then decreases as the number of firms goes up beyond nine firms. Moreover, the gray curves in Figures 3(c) and 3(d) indicate that profits and welfare decrease as well as the number of competitors goes up. The crucial distinction between this case and the previous one is the equilibrium consumer search intensity. Table 2 shows that unlike the case shown in Table 1, a relatively large share of consumers compares prices of all firms in the market. Consumers who conduct an exhaustive search in the market become disproportionately less attractive for a firm as more competitors are around. This effect, which leads to higher prices, has here a dominating influence and results in lower consumer surplus and less industry profits. As a result, welfare decreases in N as well.

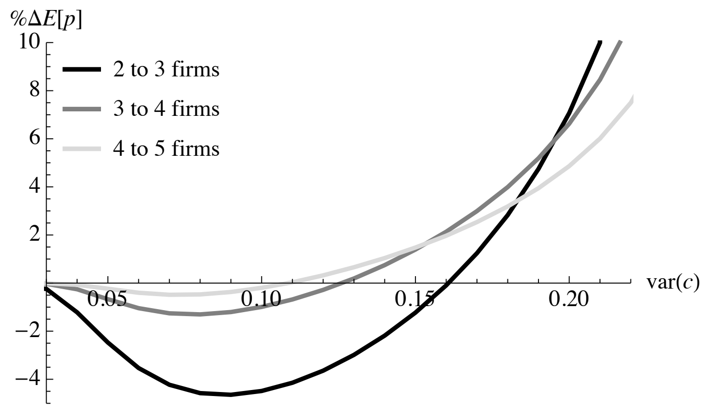


Figure 4: Effect of entry on expected price, and search cost dispersion

In the analysis above we have chosen to look at search cost distributions that reflect two relatively extreme cases (fairly high and low search cost dispersion). This has been to show as neatly as possible that the effect of an increase in the number of sellers on market prices can differ starkly. This is however a more general observation; as a matter of fact, even for intermediate levels of search cost dispersion, entry may lead to lower or higher average prices. This is illustrated in Figure 4, which shows how the level of search cost dispersion relates to the percentage change in the expected price following the entry of an additional firm.[11] The curves represent the change in expected price caused by adding a firm to a market that initially has two, three, and four firms, respectively; as the graph illustrates, for intermediate levels of search cost dispersion, say 0.12, the expected price may drop when moving from two to three firms, but may increase when moving from four to five firms. In summary, this shows that whether entry causes an increase or a decrease in average prices depends on the extent of search frictions in the market as well as the number of

[11] As before, we use the variance of a beta distribution with mean 0.5 as a measure of search cost dispersion. The percentage change in price $\%\Delta E[p]$ is defined as the change in expected price when moving from N to $N+1$ firms, divided by the expected price for N firms ($\times 100$ to make it a percentage).

firms.

4 Conclusions

The seminal contribution of Burdett and Judd (1983) has become central to the understanding of the role of information frictions in generating observed market inefficiencies. Burdett and Judd's model has seen many applications, not only in consumer search but also in labor economics and finance. This paper has generalized Burdett and Judd's (1983) model to the case in which consumers have heterogeneous search costs. This extension has been used in the empirical literature as the workhorse model for structurally estimating search costs in homogenous product markets.

Our paper has contributed to the literature by demonstrating that a symmetric Nash equilibrium in pure strategies always exists for arbitrary search cost distributions with strictly increasing CDF. In addition, we have presented a uniqueness result which, together with numerical simulations, suggest that the equilibrium will be unique at least for densities that are not too decreasing.

Finally, we have also explored how entry of firms affects the market equilibrium. We have found that the effects of an increase in the number of competitors is highly sensitive to the nature of search cost dispersion. When consumers have similar search costs, our model with search cost heterogeneity predicts effects that are in line with traditional Cournot and Betrand models, that is, mean prices fall and consumer surplus increases in the number firms. By contrast, if search costs are relatively dispersed across the consumer population, mean prices increase and consumer surplus may decrease in the number of firms.

Appendix

Proof of Proposition 1. First, suppose, on the contrary, that $\mu_1 = 0$. Then we have two possibilities: (i) either $\mu_0 = 1$ in which case the market does not open, or (ii) $\mu_k > 0$ for some $k = 2, 3, \ldots, N$ in which case all firms would charge a price equal to the marginal cost r. But if this were so, consumers would gain by deviating and searching less. Second, suppose, on the contrary, that $\mu_1 = 1$. Then firms prices would be equal to the monopoly price v. But if this were so then consumers would gain by deviating and exiting the market. Finally, suppose, on the contrary, that $1 > \mu_1 > 0$ and that $\mu_k = 0$ for all $k = 2, 3, \ldots, N$. Then $\mu_0 + \mu_1 = 1$ and the argument applied before would hold here too; as a result, there must be some $k \geq 2$ for which $\mu_k > 0$. ■

Proof of Proposition 2. Suppose, on the contrary, that firms did charge a price $\hat{p} \in (r, v]$ with strictly positive probability in equilibrium. Consider a firm i charging $\hat{p}$. The probability that $\hat{p}$ is the only price in the market is strictly positive. This occurs when all other firms are charging $\hat{p}$. From Proposition 1 we know that in equilibrium there exists some $\hat{k} \geq 2$ for which $\mu_{\hat{k}} > 0$. Consider the fraction of consumers sampling $\hat{k}$ firms. The probability that these consumers are sampling firm i is strictly positive; as a result, firm i would gain by deviating and charging $\hat{p} - \varepsilon$ since in that case the firm would attract all consumers in $\mu_{\hat{k}}$ who happened to sample firm i. This deviation would give firm i a discrete increase in its profits and thus rules out all atoms in the set $(r, v]$. It remains to be proven that an atom at the marginal cost r cannot be part of an equilibrium either. Consider a firm charging r. From Proposition 1 we know that $1 > \mu_1 > 0$. As a result, this firm would serve a fraction of consumers at least as large as μ_1/N but obtain zero profits. This implies that the firm would have an incentive to deviate by increasing its price. We now prove that the upper bound of $F(p)$ must be equal to v. Suppose not and consider a firm charging an upper bound $\overline{p} < v$. Since this firm would not sell to any consumer who compares prices, its payoff would simply be equal to $(\overline{p} - r)\mu_1/N$, which is strictly increasing in $\overline{p}$; as a result the firm would gain by deviating and charging v. ■

Proof of Theorem 1. Let $\theta := v - r$ and consider the change of variables $x_k := G(c_k)$. Then we can rewrite the equations describing the equilibrium (14)-(15) as

$$x_0 = G\left(\theta - \theta \int_0^1 \frac{x_0 - x_1}{\sum_{h=1}^N h\left(x_{h-1} - x_h\right) u^{h-1}} du\right);$$
$$x_k = G\left(\theta \int_0^1 \frac{x_0 - x_1}{\sum_{h=1}^N h\left(x_{h-1} - x_h\right) u^{h-1}} \left[k u^{k-1} - (k+1) u^k\right] du\right), k = 1, 2, \ldots, N-1, \ \left(x_N = 0\right).$$

Since $x_0 = G(c_0) > 0$ in any interesting market equilibrium, we can define $y_k = \frac{x_k}{x_0}$. Then the solution of this system will be

$$x_0 = G\left(\theta - \theta \int_0^1 \frac{1 - y_1}{1 - y_1 + \sum_{h=2}^N h\left(y_{h-1} - y_h\right) u^{h-1}} du\right),$$
$$x_1 = x_0 y_1, \ldots, x_{N-1} = x_0 y_{N-1},$$

if $y = (y_1, y_2, \ldots, y_{N-1})$ is the solution of the following system of equations:

$$y_k = \frac{G\left(\theta \int_0^1 \frac{(1 - y_1)\left[k u^{k-1} - (k+1) u^k\right]}{1 - y_1 + \sum_{h=2}^N h\left(y_{h-1} - y_h\right) u^{h-1}} du\right)}{G\left(\theta - \theta \int_0^1 \frac{1 - y_1}{1 - y_1 + \sum_{h=2}^N h\left(y_{h-1} - y_h\right) u^{h-1}} du\right)}, \quad k = 1, 2, \ldots, N-1, \ \left(y_N = 0\right). \quad (16)$$

We are looking for a solution of this latter system in $[0,1]^{N-1}$ for which $y_1 \geq y_2 \geq \ldots \geq y_{N-1}$. For this purpose, we define the set $Y = \{(y_1, y_2, \ldots, y_{N-1}) \in [0,1]^{N-1} : y_1 \geq y_2 \geq \ldots \geq y_{N-1}\}$. Likewise, define the function $H = (H_1, \ldots, H_{N-1}) : Y \setminus \{0\} \to \mathbb{R}^{N-1}$ with

$$H_k(y) = \frac{G\left(\theta \int_0^1 \frac{(1 - y_1)\left[k u^{k-1} - (k+1) u^k\right]}{1 - y_1 + \sum_{h=2}^N h\left(y_{h-1} - y_h\right) u^{h-1}} du\right)}{G\left(\theta - \theta \int_0^1 \frac{1 - y_1}{1 - y_1 + \sum_{h=2}^N h\left(y_{h-1} - y_h\right) u^{h-1}} du\right)}, \quad k = 1, 2, \ldots, N-1, \ \left(y_N = 0\right).$$

Then the solution of the system (16) is a fixed point of H. In what follows we apply Brouwer's theorem to show that the function H has a fixed point.

First we show that the function H takes values in the set Y. This is intuitively clear based on the properties of the model since by appropriate transformations it is equivalent to the inequalities $c_0 \geq c_1 \geq \cdots \geq c_{N-1}$. Here we provide a direct proof.

Lemma 1 *The function $H(\cdot)$ takes values in Y.*

Proof. Take an arbitrary $y \in Y \setminus \{0\}$. We need to prove that $0 \leq H_k(y) \leq 1$ for all $k = 1, 2, \ldots, N-1$ and $H_k(y) \leq H_{k-1}(y)$ for all $k = 2, \ldots, N-1$. The inequality $0 \leq H_k(y)$ follows straightforwardly from the nonnegativity of G. In order to prove $H_k(y) \leq 1$ and $H_k(y) \leq H_{k-1}(y)$ we use integration by parts. First we observe that

$$\int_0^1 \frac{1-y_1}{1-y_1+\sum_{h=2}^N h\left(y_{h-1}-y_h\right)u^{h-1}}du = \int_0^1 \frac{1-y_1}{1-y_1+\sum_{h=2}^N h\left(y_{h-1}-y_h\right)(1-u)^{h-1}}du.$$

By integration by parts

$$\begin{aligned}
&\int_0^1 \frac{1-y_1}{1-y_1+\sum_{h=2}^N h\left(y_{h-1}-y_h\right)(1-u)^{h-1}}du \\
&= 1-\int_0^1 \frac{(1-y_1)\,u\left[\sum_{h=2}^N h(h-1)\left(y_{h-1}-y_h\right)(1-u)^{h-2}\right]}{\left(1-y_1+\sum_{h=2}^N h\left(y_{h-1}-y_h\right)(1-u)^{h-1}\right)^2}du.
\end{aligned}$$

So the argument of G in the denominator is proportional to

$$\begin{aligned}
&1-\int_0^1 \frac{1-y_1}{1-y_1+\sum_{h=2}^N h\left(y_{h-1}-y_h\right)u^{h-1}}du \\
&= \int_0^1 \frac{(1-y_1)\,u\left[\sum_{h=2}^N h(h-1)\left(y_{h-1}-y_h\right)u^{h-2}\right]}{\left(1-y_1+\sum_{h=2}^N h\left(y_{h-1}-y_h\right)u^{h-1}\right)^2}du.
\end{aligned}$$

The argument of G in the numerator of $H_k(\cdot)$ is proportional to

$$\begin{aligned}
&\int_0^1 \frac{(1-y_1)\left[ku^{k-1}-(k+1)\,u^k\right]}{1-y_1+\sum_{h=2}^N h\,(y_{h-1}-y_h)\,u^{h-1}}du \\
&= \int_0^1 \frac{1-y_1}{1-y_1+\sum_{h=2}^N h\,(y_{h-1}-y_h)\,u^{h-1}}d\left(u^k-u^{k+1}\right) \\
&= \int_0^1 \frac{(1-y_1)\,u^k\,(1-u)\left[\sum_{h=2}^N h\,(h-1)\,(y_{h-1}-y_h)\,u^{h-2}\right]}{\left(1-y_1+\sum_{h=2}^N h\,(y_{h-1}-y_h)\,u^{h-1}\right)^2}du.
\end{aligned}$$

The inequality $H_k(y) \leq 1$ follows from the fact that $u \geq u^k(1-u)$ while the inequalities $H_k(y) \leq H_{k-1}(y)$, $k = 2, 3, \ldots, N-1$ follow because all terms in the expressions of the integrals are non-negative and u^k is decreasing in k. ■

We now apply Brouwer's fixed point theorem to prove a fixed point of H exists. Since the denominator of H_k is 0 for $y = 0$, we need to modify the function H in the neighborhood of 0. We do this in three steps: (i) we first prove that the limit inferior of H when $y \to 0$ is strictly positive (Proposition 7). (ii) We then construct a neighborhood V of 0 such that H is continuously extendable from $Y \setminus V$ to Y such that the extended function has no fixed point in V (Lemma 3, Lemma 4). (iii) Finally, we apply Brouwer's fixed point theorem to the extended function to establish the existence of a solution of the system (16).

We start by showing that the limit inferior of H is strictly positive. Since $H_k(y) \leq H_{k-1}(y)$, $k = 2, 3, \ldots, N-1$, is is sufficient to study the limit inferior of H_1.

Proposition 7 $\liminf_{\substack{y\to 0\\ y\in Y}} H_1(y) \geq \begin{cases} \frac{1}{3} & \textit{if } g(0) > 0, \\ \\ \frac{1}{9} & \textit{if } g(0) = 0 \textit{ and } g'(0) > 0. \end{cases}$

Proof. By definition $\liminf_{\substack{y\to 0\\ y\in Y}} H_1(y) = \lim_{\varepsilon\to 0} \inf\{H_1(y) : y \in Y \cap B(0,\varepsilon) \setminus \{0\}\}$, where $B(0,\varepsilon) = \left\{x \in \mathbb{R}^{N-1} : \|x\| < \varepsilon\right\}$. By Lemma 2 below there exists an $\varepsilon > 0$ such that $H_1(y)$ is increasing in y_k for $k = 2, \ldots, N-1$ on $Y \cap B(0,\varepsilon) \setminus \{0\}$. This implies that for any $y \in Y \cap B(0,\varepsilon) \setminus \{0\}$ such that $y_1 > 0$

$$H_1(y_1, y_2, \ldots, y_{N-1}) \geq H_1(y_1, y_2, \ldots, y_{N-2}, 0) \geq H_1(y_1, y_2, \ldots, 0, 0) \geq \ldots \geq H_1(y_1, 0, \ldots, 0)$$
$$= \frac{G\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)}{1-y_1+2y_1u} du\right)}{G\left(\theta - \theta \int_0^1 \frac{1-y_1}{1-y_1+2y_1u} du\right)}.$$

Therefore,

$$\liminf_{\substack{y \to 0 \\ y \in Y}} H_1(y) \geq \lim_{\varepsilon \to 0} \inf \left\{ \frac{G\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)}{1-y_1+2y_1u} du\right)}{G\left(\theta - \theta \int_0^1 \frac{1-y_1}{1-y_1+2y_1u} du\right)} : 0 < y_1 < \varepsilon \right\}.$$

The limit on the right hand side is by definition the limit inferior of $\frac{G\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)}{1-y_1+2y_1u} du\right)}{G\left(\theta - \theta \int_0^1 \frac{1-y_1}{1-y_1+2y_1u} du\right)}$ when $y_1 \to 0,\ y_1 > 0$. We show that this limit inferior is just equal to the limit, due to the fact that the limit exists. Indeed, we can apply the l'Hôpital rule to obtain

$$\lim_{\substack{y_1 \to 0 \\ y_1 > 0}} \frac{G\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)}{1-y_1+2y_1u} du\right)}{G\left(\theta - \theta \int_0^1 \frac{1-y_1}{1-y_1+2y_1u} du\right)} = \lim_{\substack{y_1 \to 0 \\ y_1 > 0}} -\frac{g\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)}{1-y_1+2y_1u} du\right) \int_0^1 \frac{u(1-2u)}{(1-y_1+2y_1u)^2} du}{g\left(\theta - \theta \int_0^1 \frac{1-y_1}{1-y_1+2y_1u} du\right) \int_0^1 \frac{u}{(1-y_1+2y_1u)^2} du}. \tag{17}$$

If $g(0) > 0$ then this limit is further equal to

$$-\frac{g\left(\theta \int_0^1 (1-2u)\, du\right) \int_0^1 u(1-2u)\, du}{g\left(\theta - \theta \int_0^1 du\right) \int_0^1 u du} = -\frac{g(0) \int_0^1 u(1-2u)\, du}{g(0) \int_0^1 u du} = -\frac{\int_0^1 u(1-2u)\, du}{\int_0^1 u du} = \frac{1}{3}.$$

If $g(0) = 0$ and $g'(0) > 0$ then the limit (17) is equal to the limit of

$$-\frac{g'\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)du}{1-y_1+2y_1u}\right) \theta \int_0^1 \frac{-2u(1-2u)du}{(1-y_1+2y_1u)^2} \int_0^1 \frac{u(1-2u)du}{(1-y_1+2y_1u)^2} + g\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)du}{1-y_1+2y_1u}\right) \int_0^1 \frac{2u(1-2u)^2 du}{(2uy_1-y_1+1)^3}}{g'\left(\theta - \theta \int_0^1 \frac{(1-y_1)du}{1-y_1+2y_1u}\right) \int_0^1 \frac{(-\theta)(-2u)du}{(1-y_1+2y_1u)^2} \int_0^1 \frac{udu}{(1-y_1+2y_1u)^2} + g\left(\theta - \theta \int_0^1 \frac{(1-y_1)du}{1-y_1+2y_1u}\right) \int_0^1 \frac{2u(1-2u)du}{(2uy_1-y_1+1)^3}}$$
$$= -\frac{g'(0)\,\theta \int_0^1 (-2u)(1-2u)\, du \int_0^1 u(1-2u)\, du + g(0) \int_0^1 2u(1-2u)^2\, du}{g'(0)(-\theta) \int_0^1 (-2u)\, du \int_0^1 u du + g(0) \int_0^1 2u(1-2u)\, du}$$
$$= \frac{\int_0^1 (-2u)(1-2u)\, du \int_0^1 u(1-2u)\, du}{\int_0^1 (-2u)\, du \int_0^1 u du} = \frac{1}{9}.$$

■

Lemma 2 *There exists an $\varepsilon > 0$ such that $H_1(y)$ is increasing in y_k for $k = 2, \ldots, N-1$ on*

$Y \cap B(0,\varepsilon) \setminus \{0\}$.

Proof. For simplicity of notation we use

$$H_1(y) = \frac{U(y)}{D(y)},$$

where $U, D : Y \to \mathbb{R}$

$$U(y) = G\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)}{1-y_1+\sum_{h=2}^N h(y_{h-1}-y_h)u^{h-1}} du\right),$$
$$D(y) = G\left(\theta - \theta \int_0^1 \frac{1-y_1}{1-y_1+\sum_{h=2}^N h(y_{h-1}-y_h)u^{h-1}} du\right).$$

The partial derivatives of U and D with respect to y_k for some $k \in \{2,\ldots,N-1\}$ are

$$\frac{\partial U}{\partial y_k} = g\left(\theta \int_0^1 \frac{(1-y_1)(1-2u)}{1-y_1+\sum_{h=2}^N h(y_{h-1}-y_h)u^{h-1}} du\right)\theta I_U(y),$$
$$\frac{\partial D}{\partial y_k} = g\left(\theta - \theta \int_0^1 \frac{1-y_1}{1-y_1+\sum_{h=2}^N h(y_{h-1}-y_h)u^{h-1}} du\right)(-\theta) I_D(y),$$

where

$$I_U(y) = \int_0^1 \frac{(1-y_1)(1-2u)\left[ku^{k-1}-(k+1)u^k\right]}{\left(1-y_1+\sum_{h=2}^N h(y_{h-1}-y_h)u^{h-1}\right)^2} du,$$
$$I_D(y) = \int_0^1 \frac{(1-y_1)\left[ku^{k-1}-(k+1)u^k\right]}{\left(1-y_1+\sum_{h=2}^N h(y_{h-1}-y_h)u^{h-1}\right)^2} du.$$

By integration by parts

$$I_D(y) = 2\int_0^1 (1-y_1)\left(u^k-u^{k+1}\right) \frac{\sum_{h=2}^N h(h-1)(y_{h-1}-y_h)u^{h-2}}{\left(1-y_1+\sum_{h=2}^N h(y_{h-1}-y_h)u^{h-1}\right)^3} du.$$

Now, $I_D \geq 0$ for any $y \in Y$ because all terms in the integral are nonnegative. Therefore $\frac{\partial D}{\partial y_k} \leq 0$ for any $y \in Y$, which implies that D is decreasing in y_k at any point $y \in Y$.

Regarding the integral I_U we note that

$$I_U(0) = \int_0^1 (1-2u)\left[ku^{k-1} - (k+1)\,u^k\right] du = \frac{2}{(k+1)(k+2)} > 0.$$

So for each k there is an $\varepsilon_k > 0$ such that $I_U(y) \geq 0$ for any $y \in Y \cap B(0, \varepsilon_k)$; so for $\varepsilon = \min\{\varepsilon_2, \ldots, \varepsilon_{N-1}\}$ it holds that $I_U(y) \geq 0$ for any $y \in Y \cap B(0, \varepsilon)$. Therefore $\frac{\partial U}{\partial y_k} \geq 0$ for any $y \in Y \cap B(0, \varepsilon)$ and $k = 2, \ldots, N-1$. This implies that U is increasing in y_k for any $y \in Y \cap B(0, \varepsilon)$. This establishes that $H_1(y)$ is increasing in y_k for any $y \in Y \cap B(0, \varepsilon) \setminus \{0\}$. ■

So, we have established that the limit inferior of $H_1(y)$ when $y \to 0$ is strictly positive. Then the following statement establishes that there is an $\varepsilon > 0$ such that the set $Y \cap [0, \varepsilon]^{N-1}$ can take the role of the neighborhood V mentioned above.

Lemma 3 *Let $H : Y \setminus \{0\} \to \mathbb{R}^{N-1}$ be a continuous function such that $\liminf_{\substack{y \to 0 \\ y \in Y}} H_1(y) \geq a > 0$. Then there exists $\varepsilon > 0$ such that $H_1(y) > \varepsilon$ for any $y = (y_1, y_2, \ldots, y_{N-1}) \in Y \setminus \{0\}$ with $y_1 \leq \varepsilon$.*

Proof. Condition $\liminf_{\substack{y \to 0 \\ y \in Y}} H_1(y) \geq a > 0$ implies that for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that $H_1(y) > a - \delta$ for any $y = (y_1, y_2, \ldots, y_{N-1}) \in Y \setminus \{0\}$ with $y_1 \leq \varepsilon_\delta$. Take $\delta_1 > 0$ such that $a - \delta_1 > 0$. Then there exists $\varepsilon_1 > 0$ such that $H_1(y) > a - \delta_1$ for any $y = (y_1, y_2, \ldots, y_{N-1}) \in Y \setminus \{0\}$ with $y_1 \leq \varepsilon_1$. Now, if $a - \delta_1 > \varepsilon_1$ then choose $\varepsilon = \varepsilon_1$ and the result is proved. If $a - \delta_1 \leq \varepsilon_1$ then choose $\varepsilon > 0$ such that $a - \delta_1 > \varepsilon$. For any $y = (y_1, y_2, \ldots, y_{N-1}) \in Y \setminus \{0\}$ with $y_1 \leq \varepsilon < \varepsilon_1$ it holds that $H_1(y) > a - \delta_1 > \varepsilon$, so in this case the result is proved as well. ■

Since we established condition $\liminf_{\substack{y \to 0 \\ y \in Y}} H_1(y) \geq a > 0$ in Proposition 7 we can now use ε from Lemma 3. Define the function $J = (J_1, \ldots, J_{N-1}) : Y \to \mathbb{R}^{N-1}$ such that

$$J(y) = \begin{cases} H(y) & \text{for } y \in Y \setminus Y_\varepsilon, \\ \\ H(\varepsilon, y_2, \ldots, y_{N-1}) & \text{for } y \in Y_\varepsilon, \end{cases}$$

where $Y_\varepsilon = \{(y_1, y_2, \ldots, y_{N-1}) \in Y : y_1 \leq \varepsilon\} = Y \cap [0, \varepsilon]^{N-1}$. Notice that J is also defined in 0.

Lemma 4 *The function J has the properties: (i) J is continuous. (ii) J takes values in Y. (iii) J has no fixed point in Y_ε.*

Proof. (i) Based on the fact that H is continuous, J is also continuous at points y that are not on the boundary between Y_ε and $Y \setminus Y_\varepsilon$. The only non-trivial case is when y is on the boundary between Y_ε and $Y \setminus Y_\varepsilon$, that is, in $\{(y_1, y_2, \ldots, y_{N-1}) \in Y : y_1 = \varepsilon\}$. In this case the limit of $J(t^n)$ for a sequence $(t^n)_{n\geq 1} \subset \{(y_1, y_2, \ldots, y_{N-1}) \in Y : y_1 > \varepsilon\}$ with $t^n \to y$ should be $J(y)$. Indeed, $J(t^n) = H(t^n) \to H(y) = H(\varepsilon, y_2, \ldots, y_{N-1}) = J(y)$.

(ii) The fact that J takes values in Y follows from Lemma 1 trivially for the case $(y_1, y_2, \ldots, y_{N-1}) \in Y \setminus Y_\varepsilon$. For the case $(y_1, y_2, \ldots, y_{N-1}) \in Y_\varepsilon$ it follows because $(\varepsilon, y_2, \ldots, y_{N-1}) \in Y$ for any $(y_1, y_2, \ldots, y_{N-1}) \in Y_\varepsilon$, so $J(\varepsilon, y_2, \ldots, y_{N-1}) = H(\varepsilon, y_2, \ldots, y_{N-1}) \in Y$.

(iii) For an arbitrary $(y_1, y_2, \ldots, y_{N-1}) \in Y_\varepsilon$ we have $J_1(y_1, y_2, \ldots, y_{N-1}) = H_1(\varepsilon, y_2, \ldots, y_{N-1})$. Since $y = (\varepsilon, y_2, \ldots, y_{N-1}) \in Y \setminus \{0\}$ with $y_1 \leq \varepsilon$, by Lemma 3 it holds that $H_1(\varepsilon, y_2, \ldots, y_{N-1}) > \varepsilon$. Thus $J_1(y_1, y_2, \ldots, y_{N-1}) > \varepsilon \geq y_1$, so $(y_1, y_2, \ldots, y_{N-1})$ cannot be a fixed point of J. ■

Finally we can establish that the system of equations (16) has a solution. By Lemma 4 the function $J : Y \to Y$ is continuous. Y is a convex and compact set, so by Brouwer's fixed point theorem J has a fixed point y^*. The fixed point cannot be in Y_ε by Lemma 4, so $y^* \in Y \setminus Y_\varepsilon$. Therefore $y^* = J(y^*) = H(y^*)$, that is, $y^* \in Y \setminus Y_\varepsilon$ is a fixed point of H. By definition, any fixed point of H is a solution of the system (16). This completes the proof of existence of equilibrium in Theorem 1. ■

Proof of Proposition 6. Setting $N = 2$ in equations (16) gives

$$
\begin{aligned}
x_0 &= G\left(\theta - \theta \int_0^1 \frac{x_0 - x_1}{x_0 - x_1 + 2x_1 u} du\right); \\
x_1 &= G\left(\theta \int_0^1 \frac{(x_0 - x_1)(1 - 2u)}{x_0 - x_1 + 2x_1 u} du\right).
\end{aligned}
$$

Using the notation introduced before, $y = x_1/x_0 \in (0, 1)$, the solution to this system of equations is given by the solution to $H_1(y) - y = 0$, or

$$\phi(y) \equiv yG(\theta - \theta(1-y)I(y)) - G(\theta(1-y)J(y)) = 0. \tag{18}$$

where

$$I(y) = \int_0^1 \frac{1}{1-y+2yu}du = \frac{\log(1+y) - \log(1-y)}{2y};$$
$$J(y) = \int_0^1 \frac{1-2u}{1-y+2yu}du = \frac{\log(1+y) - \log(1-y) - 2y}{2y^2}.$$

Let $G(c) = (c/\overline{c})^a$ for some $a > 0$ with support $[0, \overline{c}]$. From equation (18), since the case $y = 0$ is not interesting and $G(c_0(y)) > 0$ for $y > 0$, it is sufficient to prove that the equation

$$y = \frac{G(c_1(y))}{G(c_0(y))} \tag{19}$$

has a unique solution. Since the LHS of equation (19) is increasing in y, it suffices to show that the RHS decreases in y.[12] Let $h(y)$ denote the RHS of equation (19):

$$h(y) = \frac{\left(\frac{c_1(y)}{\overline{c}}\right)^a}{\left(\frac{c_0(y)}{\overline{c}}\right)^a} = \frac{c_1(y)^a}{c_0(y)^a}$$

The derivative of $h(y)$ is

$$\begin{aligned}\frac{dh(y)}{dy} &= \frac{a\frac{dc_1(y)}{dy}c_1^{a-1}(y)\,c_0^a(y) - ac_1^a(y)\frac{dc_0(y)}{dy}c_0^{a-1}(y)}{c_0^{2a}(y)} \\ &= \frac{ac_1^{a-1}(y)\,c_0^{a-1}(y)}{c_0^{2a}(y)}\left(\frac{dc_1(y)}{dy}c_0(y) - c_1(y)\frac{dc_0(y)}{dy}\right).\end{aligned}$$

[12] In general this is equivalent to requiring that

$$\frac{g(c_1(y))}{G(c_1(y)}\frac{dc_1(y)}{dy} - \frac{g(c_0(y))}{G(c_0(y)}\frac{dc_0(y)}{dy} < 0.$$

Intuitively one case when this is violated can occur when the density g decreases steeply so that $g(c_1(y))$ is much bigger than $g(c_0(y)$. With not too decreasing densities, the inequality above will tend to be satisfied.

Since

$$\frac{dc_1(y)}{dy} = \frac{2y(2+y) - (1+y)(2-y)\ln\frac{1+y}{1-y}}{2y^3(1+y)},$$
$$\frac{dc_0(y)}{dy} = \frac{-2y + (1+y)\ln\frac{1+y}{1-y}}{2y^2(1+y)},$$

we obtain that

$$\frac{dc_1(y)}{dy}c_0(y) - c_1(y)\frac{dc_0(y)}{dy} =$$
$$= 4y^2(1+2y) + 2y(1+y)(2-y)\ln\frac{1-y}{1+y} + \left(1-y^2\right)(1-y)\ln^2\frac{1-y}{1+y}.$$

This expression is negative for $0 < y < 1$, so $dh(y)/dy < 0$, and therefore, the equilibrium is unique. ■

References

Acemoglu, D., Shimer, R., 2000. Wage and technology dispersion. Review of Economic Studies 67, 585-607.

Anderson, S.P., Renault, R., 1999. Pricing, product diversity, and search costs: a Bertrand-Chamberlin-Diamond model. RAND Journal of Economics 30, 719-735.

Barron, J.M., Taylor, B.A., Umbeck, J.R., 2004. Number of sellers, average prices, and price dispersion. International Journal of Industrial Organization 22, 1041-1066.

Baye, M.R., Morgan, J., Scholten, P., 2004. Price dispersion in the small and in the large: evidence from an Internet price comparison site. Journal of Industrial Economics 52, 463-496.

Blevins, J.R., Senney, G.T., 2016. Dynamic selection and distributional bounds on search costs in dynamic unit-demand models. Working paper.

Burdett, K., Judd, K.L., 1983. Equilibrium price dispersion. Econometrica 51, 955-969.

Burdett, K., Mortensen, D.T., 1998. Wage differentials, employer size, and unemployment. International Economic Review 39, 257-273.

Chen, Y., Riordan, M.H., 2008. Price-increasing competition. RAND Journal of Economics 39, 1042-1058.

Chen, Y., Zhang, T., 2016. Entry and welfare in search markets. Working paper.

De los Santos, B., 2012. Consumer search on the Internet. Working paper.

Fershtman, C., Fishman, A., 1992. Price cycles and booms: dynamic search equilibrium. American Economic Review 82, 1221-1233.

González, X., Miles, D., 2015. Price dispersion and supermarket heterogeneity in Spanish food retailing. Working paper.

Haynes, M., Thompson, S., 2008. Price, price dispersion and number of sellers at a low entry cost shopbot. International Journal of Industrial Organization 26, 459-472.

Hong, H., Shum, M., 2006. Using price distributions to estimate search costs. RAND Journal of Economics 37, 257-275.

Janssen, M.C.W., Moraga-González, J.L., 2004. Strategic pricing, consumer search and the number of firms. Review of Economic Studies 71, 1089-1118.

Lach, S., Moraga-González, J.L., 2009. Asymmetric price effects of competition. Working paper.

Lewis, M., 2008. Price dispersion and competition with differentiated sellers. Journal of Industrial Economics 56, 654-678.

MacMinn, R.D., 1980. Search and market equilibrium. Journal of Political Economy 88, 308-327.

Miessi Sanches, F.A., Silva Junior, D., Srisuma, S., 2015. Minimum distance estimation of search costs using price distribution. Working paper.

Moraga-González, J.L., Sándor, Z., Wildenbeest, M.R., 2013. Semi-nonparametric estimation of consumer search costs. Journal of Applied Econometrics 28, 1205-1223.

Moraga-González, J.L., Sándor, Z., Wildenbeest, M.R., 2015. Prices and heterogeneous search costs. Forthcoming in RAND Journal of Economics.

Moraga-González, J.L., Wildenbeest, M.R., 2008. Maximum likelihood estimation of search costs. European Economic Review 52, 820-848.

McAfee, R.P., 1995. Multiproduct equilibrium price dispersion. Journal of Economic Theory 67, 83-105.

Nishida, M., Remer, M., 2015. The determinants and consequences of search cost heterogeneity: evidence from local gasoline markets. Working paper.

Perloff, J.M., Salop, S.C., 1985. Equilibrium with product differentiation. Review of Economic Studies 52, 107-120.

Richards, T.J., Hamilton, S.F., Allender, W., 2016. Search and price dispersion in online grocery markets. International Journal of Industrial Organization 47, 255-281.

Salop, S.C., 1979. Monopolistic competition with outside goods. The Bell Journal of Economics 10, 141-156.

Seade, J., 1980. On the effects of entry. Econometrica 48, 479-489.

Stahl, D.O., 1989. Oligopolistic pricing with sequential consumer search. American Economic Review 79, 700-712.

Varian, H., 1980. A model of sales. American Economic Review 70, 651-659.

Wildenbeest, M.R., 2011. An empirical model of search with vertically differentiated products. RAND Journal of Economics 42, 729-757.

Zhang, X., Chan, T.Y., Xie, Y., 2015. Price search and periodic price discounts. Working paper.